MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. XXXVI.

MATHEMATICAL QUESTIONS

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SOLUTIONS,

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WITH MANY

Papers and Solutions not published in the "Educational Times."

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR

OF THE

GENERAL MEDICAL COUNCIL.

VOL. XXXVI.

LONDON:

C. F. HODGSON & SON, GOUGH SQUARE,

FLEET STREET.

1881.

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Mathematical Papers, &c.

Solved Questions.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

6470. (By Prof. CAYLEY, F.R.S.)—It is required, by a real or imaginary linear transformation, to express the equation of a given cubic curve in the form $xy - z^2 = \{(z^2-x^2)(z^2-k^2x^2)\}^{\frac{1}{2}}$.

Solution by W. J. C. SHARP, M.A.

I have shown (*Quarterly Journal*, Vol. XVI., p. 186), that the equation to a proper cubic can always be reduced to the form

$$a\xi^3 + 3c_1\xi\zeta^2 + c\zeta^3 - 3b_3\eta^2\zeta = 0, \quad \text{that is,} \quad \xi^3 + \mu\xi\zeta^2 + \zeta^3 + \eta^2\zeta = 0.$$

Comparing this with the proposed form,

$$xy^2 - 2yz^2 + (1+k^2)\,xz^2 - k^2x^3 = 0,$$

the transformation may be effected by putting

$$\zeta = \nu^2\{(1+k^2)\,x - 2y\}, \quad \nu\eta = z, \quad \xi = \mathrm{A}x + \mathrm{B}y,$$

where the four unknowns A, B, k^2, ν are to be determined by substituting, for ξ and ζ in $(k^2x^2 - y^2)\,x = \xi^3 + \mu\xi\zeta^2 + \zeta^3$, and equating coefficients.

6663. (By Prof. WOLSTENHOLME, M.A.)—If p, q be the lengths of two tangents drawn to an ellipse $a^2y^2 + b^2x^2 = a^2b^2$ from any point O of the ellipse $a^2y^2 + b^2x^2 = m^2a^2b^2$, prove that

(1) $$\left\{\frac{p^2+q^2}{a^2+b^2} - 4\left(1-\frac{1}{m^2}\right)^2\right\}^2\left(\frac{a^2+b^2}{2ab}\right)^2 = \left(\frac{2}{m^2}-1\right)^2\left(\frac{p^2q^2}{a^2b^2} - \frac{4(m^2-1)^3}{m^4}\right);$$

(2) when $m^2 < (a^2+b^2)\,a^{-2}$, then pq will have a minimum value when the point O is at the ends of the major axis, and a maximum when at the ends

of the minor; (3) when $m^2 > (a^2+b^2)\,a^{-1}$ and $< (a^2+b^2)\,b^{-2}$, then pq will have maxima values when O is at any vertex and minima values (all equal) when the two tangents are at right angles; and (4) when $m^2 > (a^2+b^2)\,b^{-2}$ the maximum values will be when O is at the ends of the major axis and the minimum when at the ends of the minor, while there are no other maxima nor minima.

Solution by G. Heppel, M.A.; A. Martin, M.A.; *and others.*

1. Let O be $x = ma\cos\theta,\ y = mb\sin\theta$; then chord of contact is

$$mb\cos\theta \,.\, x + ma\sin\theta \,.\, y = ab,$$

and the points of contact are

$$mx = a\cos\theta \pm (m^2-1)^{\frac{1}{2}}\,a\sin\theta,\quad my = b\sin\theta \mp (m^2-1)^{\frac{1}{2}}\,b\cos\theta.$$

If we put $m = \sec\phi$, we get $x = a\cos(\theta\mp\phi)$, $y = b\sin(\theta\mp\phi)$; from which, after simplifying and reducing, we obtain

$$2p^2\cot^2\phi = a^2+b^2-(a^2-b^2)\cos 2(\theta-\phi),$$

$$2q^2\cot^2\phi = a^2+b^2-(a^2-b^2)\cos 2(\theta+\phi);$$

therefore, by adding and subtracting,

$$(a^2-b^2)\cos 2\theta = \frac{a^2+b^2-(p^2+q^2)\cot^2\phi}{\cos 2\phi};\quad (a^2-b^2)\sin 2\theta = -\frac{(p^2-q^2)\cot^2\phi}{\sin 2\phi};$$

whence, eliminating θ, and replacing functions of ϕ by functions of m,

$$\left\{\frac{p^2+q^2}{a^2+b^2}-4\left(1-\frac{1}{m^2}\right)^2\right\}^2\left(\frac{a^2+b^2}{2ab}\right)^2 = \left(\frac{2}{m^2}-1\right)^2\left(\frac{p^2q^2}{a^2b^2}-4\,\frac{(m^2-1)^3}{m^4}\right).$$

2. We have $u = 4p^2q^2 = \tan^4\phi\,\{(a^2+b^2)^2-2\,(a^4-b^4)\cos 2\theta\cos 2\phi +$
$\frac{1}{2}(a^2-b^2)^2(\cos 4\theta+\cos 4\phi)\}$;

therefore $\quad \dfrac{du}{d\theta} = \tan^4\phi\,\{4\,(a^4-b^4)\cos 2\phi\sin 2\theta - 2\,(a^2-b^2)^2\sin 4\theta\},$

$$\frac{d^2u}{d\theta^2} = \tan^4\phi\,\{8\,(a^4-b^4)\cos 2\phi\cos 2\theta - 8\,(a^2-b^2)^2\cos 4\theta\}.$$

Now $\dfrac{du}{d\theta} = 0$ in three cases,—(i.) when $\theta = 0$ or π, (ii.) when $\theta = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$, (iii.) when $(a^2+b^2)\cos 2\phi = (a^2-b^2)\cos 2\theta$.

In case (i.) O is at the end of the major axis, and the examination of the sign of $\dfrac{d^2u}{d\theta^2}$ shows that u is a minimum from $m = 1$ to $m = (a^2+b^2)a^{-2}$, and a maximum from $m = (a^2+b^2)a^{-2}$ to $m = \infty$.

In case (ii.) O is at the end of the minor axis, and u is a maximum from $m = 1$ to $m = (a^2+b^2)b^{-2}$, and a minimum from $m = (a^2+b^2)b^{-2}$ to $m = \infty$.

In case (iii.) the sign of $\dfrac{d^2u}{d\theta^2}$ depends upon that of

$$(a^2+b^2)\cos 2\phi\cos 2\theta - (a^2-b^2)\cos 4\theta,$$

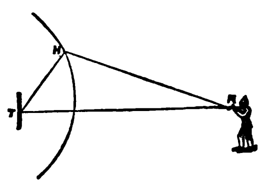
or on that of $(a^2-b^2)\,\{\cos^2 2\theta - \cos 4\theta\}$, or on that of $\sin^2 2\theta$; therefore there is a minimum. And in this case we have

$$a^2(\cos 2\theta - \cos 2\phi) = b^2(\cos 2\theta + \cos 2\phi),$$
$$a^2 \sin(\theta+\phi)\sin(\phi-\theta) = b^2 \cos(\theta+\phi)\cos(\theta-\phi).$$

Comparing this with the equations of the points of contact, we see that it gives the condition that the two tangents are at right angles. Lastly, as the locus of the intersection of pairs of tangents at right angles is the circle $x^2+y^2 = a^2+b^2$, we infer, first, that the limits of m to render such tangents possible are from $(a^2+b^2)a^{-2}$ to $(a^2+b^2)b^{-}$; and secondly, that all the four minima for any value of m are equal, because the tangents are drawn from points symmetrically situated.

6636. (By LIZZIE A. KITTUDGE.)—On a plane field, the *crack* of the rifle and the *thud* of the ball striking the target are heard at the same instant; find the locus of the hearer.

Solution by C. MORGAN, M.A.; W. M. COATES, B.A.; *and others.*



Let r_1, r_2 be the respective distances of the hearer (H) from the rifleman (R), and the target (T); v the velocity of sound in air (assuming that the sounds arising from a *crack* and *thud* travel through air with equal velocity); and t the time the ball takes to reach the target; then

$$\frac{r_1}{v} = \frac{r_2}{v} + t, \quad \text{or} \quad r_1 - r_2 = vt;$$

hence the locus of the hearer (H) is an hyperbola whose foci are R and T.

6517. (By J. J. WALKER, M.A.)—The points of intersection of two conics $ax^2 \ldots = 0$, $a'x^2 \ldots = 0$ are given by,

$$(C_1y^2 - 2F_1yz + B_1z^2)^2 - 4(Cy^2 - 2Fyz + Bz^2)(C'y^2\ldots) = 0,$$

and two similar equations; $C_1\ldots$, $C\ldots$, $C'\ldots$ being coefficients of the contravariant conics of the system, viz., $C_1 = ab' + a'b - 2ff'\ldots$

Solution by the PROPOSER.

The invariant (I) of $S \equiv ax^2 + 2(hy+gz)x + by^2 + 2fyz + cz^2$, considered as a quadratic in x is $I = a(by^2 + 2fyz + cz^2) - (hy+gz)^2 = Cy^2 - 2Fyz + By^2$, where $C = ab - h^2\ldots$ Similarly the invariant (I′) of $S' \equiv a'x^2 + \ldots$ Also the invariant of the system S, S′, considered as two quadratic forms in x,

is $J = a\,(b'y^2 + 2f'yz + c'z^2) + a'\,(by^2 + 2fyz + cz^2) - 2\,(hy + gz)\,(h'y + g'z)$
$= C_1 y^2 - 2F_1 yz + B_1 y^2$,
where $C_1 = ab' + a'b - 2hh' \ldots$ Now the resultant of the two quadratics is equal to $J^2 - 4II'$. Hence, to determine the common $y : z$, we have the equation given in the Question.

6643. (By A. STEIN, Ph.D.)—A mark is made on a vertical tower at a known height from a horizontal plane, the altitude of this mark and of the top of the tower is observed from a point in the plane: find the probable error of the height of the tower deduced from these angles in terms of the probable error in the measurement of an angle; and show that the best position of observation is that in which the sum of the two altitudes is a right angle.

Solution by Prof. MATZ, M.A.; *the* Rev. T. R. TERRY, M.A.; *and others.*

Let a be the height of the mark, b of the tower, and ϕ and θ the angles subtended; then $b = a \tan\theta \cot\phi$. If δb be the error consequent on an error dx in measuring an angle, supposed the same in each case,

$$b + \delta b = a\left(\tan\theta + \frac{\delta x}{\cos^2\theta}\right)\left(\cot\phi - \frac{\delta x}{\sin^2\phi}\right), \quad \therefore\ \delta b = \tfrac{1}{2}a\,\frac{\sin 2\phi - \sin 2\theta}{\sin^2\phi\,\cos^2\theta}\,\delta x.$$

Therefore the best position is given by $\sin 2\phi = \sin 2\theta$, or $\theta + \phi = \frac{1}{2}\pi$.

6469. (By Professor SYLVESTER, F.R.S.)—Let q be any integer containing only n distinct prime factors $p, p_2, p_3 \ldots p_n$, and let the n quantities $2^i - 1,\ 2^{i_2} - 1,\ 2^{i_3}, \ldots\ 2^{i_n} - 1$ (where $i, i_2, \ldots i_n$ are any n integers) also only contain among them the same n prime factors; show that a rule may be given for extracting the square root of any number by means of continual multiplication or division and extraction of square root only. Apply this principle to giving rules for the extraction of all roots up to the 20th by means of the above named processes.

Solution by W. J. C. SHARP, M.A.

Any fraction whatever, $p : q$ say, may be expressed either as a terminating or a recurring radix fraction in the scale of 2.

If $\dfrac{p}{q} = \dfrac{1}{2^a} + \dfrac{1}{2^b} + \dfrac{1}{2^c} + \&c.$, where a, b, &c. are in ascending order of magnitude, we have $n^{\frac{p}{q}} = n^{\frac{1}{2^a}} \times n^{\frac{1}{2^b}} \times n^{\frac{1}{2^c}} + \&c.$,
since the only digits in this scale are 0 and 1. Prof. SYLVESTER [in the annexed *Note*], gives $n^{\frac{1}{3}} = n^{\frac{1}{4} + \frac{1}{16} + \&c.}$ Now $\frac{1}{3} = \cdot\dot{0}\dot{1}$ in the scale of two, which leads to the same result. His other examples may also be deduced in the same way.

As Prof. SYLVESTER anticipates, a similar statement will hold if any number a be substituted for 2; but as the digits are more numerous in other scales, involution may be required.

[In elucidation of the theorem, Prof. SYLVESTER adds the following note:—"To understand it, suppose you want the cube root of N: this is

$$N^{\frac{1}{3}} = N^{\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\dots} = N^{\frac{1}{4}}.N^{\frac{1}{16}}.N^{\frac{1}{64}}\dots$$

Again, if you want the 5th root,

$$\frac{1}{16}+\frac{1}{16^2}+\frac{1}{16^3}+\dots=\frac{1}{15},\quad \frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\dots=\frac{1}{3};$$

therefore $$\frac{1}{5}=\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\dots-2\left(\frac{1}{16}+\frac{1}{16^2}+\frac{1}{16^3}+\dots\right).$$

Here $$N^{\frac{1}{5}} = N^{\frac{1}{4}}.N^{-\frac{1}{16}}.N^{\frac{1}{64}}.N^{-\frac{1}{256}}\dots$$

So, again, $\dfrac{1}{3^2}=\dfrac{1}{4}+\dfrac{2}{4^2}+\dfrac{3}{4^3}+\dots$, therefore $N^{\frac{1}{9}} = N^{\frac{1}{4}}.N^{\frac{1}{8}}.(N^{\frac{1}{64}})^3\dots$

And so in general. I presume that it is true in general (as it certainly is for all the cases up to 20) that we may always find n prime factors (n being some finite number), and n numbers i such that 2^i-1 shall only contain the n factors in question; probably also the i's may all be taken primes. Moreover, any number, or at least any prime number a, no doubt may be substituted for 2."]

6601. (By Prof. WOLSTENHOLME, M.A.)—Prove that any quadric surface $u=0$ is its own polar reciprocal with respect to any surface whose equation is $2uu_0=\left(x_0\dfrac{du}{dx}+y_0\dfrac{du}{dy}+z_0\dfrac{du}{dz}+w_0\dfrac{du}{dw}\right)^2$, u being a homogeneous function of x, y, z, w, and u_0 the same function of x_0, y_0, z_0, w_0.

Solution by G. F. WALKER, M.A.; CHRISTINE LADD; *and others.*

If (x', y', z', w') be any point on $u=0$, the pole of the tangent plane at (x', y', z', w') with respect to $2u_0u=\left(x_0\dfrac{du}{dx}\dots\right)^2$ is given by the equations

$$\left\{-u_0\frac{du}{dx}+\left(x_0\frac{du}{dx}\dots\right)\frac{du_0}{dx_0}\right\}\div\frac{du'}{x'}=\text{\&c.}=\text{\&c.};$$

each of these equal quantities is easily* found to be equal to

$$\left\{-u_0x+\left(x_0\frac{du}{dx}\dots\right)x_0\right\}\frac{1}{x'}=\text{\&c.}=\text{\&c.},$$

* Prof. WOLSTENHOLME remarks that, whenever he sees the word "easily" so used, he always interprets it to mean "with some difficulty."

and therefore the required polar reciprocal is given by

$$\left\{-u_0\frac{du}{dx}+\left(x_0\frac{du}{dx}\cdots\right)\frac{du_0}{dx_0}\right\}\left\{-u_0x+\left(x_0\frac{du}{dx}\cdots\right)x_0\right\}\cdots=0,$$

$$2u_0^2u-u_0\left(x_0\frac{du}{dx}\cdots\right)2\left(x_0\frac{du}{dx}\cdots\right)+\left(x_0\frac{du}{dx}\cdots\right)^2 2u_0=0,$$

which is $u=0$.

6666. (By Professor GENESE, M.A.)—If P be a point on a rectangular hyperbola, AA′ any diameter; the tangent to the curve at P, and the tangent to the circle APA′ at P, divide AA′ internally and externally in the duplicate of the ratio AP : A′P.

Solution by R. KNOWLES, B.A., L.C.P.; G. HEPPEL, M.A.; *and others.*

Take the ordinary axes, let the point P be (h, k), and let the tangents to the hyperbola and circle cut the major axis in T, Q respectively. Let $CT = x_1$, $CQ = x_2$. Then the equation to the tangent to the hyperbola is $hx-ky=a^2$, hence $CT=\dfrac{a^2}{h}$. Also $QP^2=QA\,.\,QA'$, $(x_2-h)^2+k^2=x_2^2-a^2$; and from the equation to the hyperbola $k^2=h^2-a^2$; therefore $x_2=h$;

hence we have $\qquad \dfrac{AT}{A_1T}=\dfrac{ah+a^2}{ah-a^2}=\dfrac{h+a}{h-a},\quad \dfrac{AQ}{A_1Q}=\dfrac{h+a}{h-a},$

and $\qquad \dfrac{AP^2}{A_1P^2}=\dfrac{k^2+(h+a)^2}{k^2+(h-a)^2}=\dfrac{2h^2+2ah}{2h^2-2ah}=\dfrac{h+a}{h-a}.$

[The tangent to the circle at P is perpendicular to the major axis.]

6714. (By J. J. WALKER, M.A.)—Given two sides b, c of a spherical triangle ABC, of which the angle A is equal to the sum of the angles B, C; show that these angles are determined by

$$\cos A=-\tan\tfrac{1}{2}b\tan\tfrac{1}{2}c \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\sin\tfrac{1}{2}b\cot B=\sin\tfrac{1}{2}c\cos^{\frac{1}{2}}\tfrac{1}{2}(b+c)\cos^{\frac{1}{2}}\tfrac{1}{2}(b-c)\quad\ldots\ldots\ldots\ldots(2).$$

Solution by R. TUCKER, M.A.; Prof. MATZ, M.A.; *and others.*

Let D be the mid-point of BC, and take $AD=p=BD=CD$; then

$$\tan\tfrac{1}{2}c=\tan p\cos B,\quad \tan\tfrac{1}{2}b=\tan p\cos C\quad\ldots\ldots\ldots\ldots(a).$$

Now $\cos B\cos C-\sin B\sin C=\cos A=-\cos B\cos C+\sin B\sin C\cos 2p$,

whence $\quad \cos A=-\cos B\cos C\tan^2 p=-\tan\tfrac{1}{2}b\tan\tfrac{1}{2}c$, by (a).

Or thus, by Napier's Analogies,

$$\tan\tfrac{1}{2}A = \frac{\cos\frac{1}{2}(b-c)}{\cos\frac{1}{2}(b+c)}\cot\tfrac{1}{2}(B+C) = \frac{\cos\frac{1}{2}(b-c)}{\cos\frac{1}{2}(b+c)}\cot\tfrac{1}{2}A\,;$$

therefore $\tan^2\frac{1}{2}A = \dfrac{\cos\frac{1}{2}(b-c)}{\cos\frac{1}{2}(b+c)}$, whence $\cos A = -\tan\frac{1}{2}b\tan\frac{1}{2}c$.

Again, $\cos b + \cos c = 2\cos^2 p$, $\therefore\ \cos p = \cos^{\frac{1}{2}}\frac{1}{2}(b+c)\cos^{\frac{1}{2}}\frac{1}{2}(b-c)$...($\beta$);

$$\cos p = \cot B\cot\tfrac{1}{2}q,\quad \cos p = \cot C\tan\tfrac{1}{2}q,\quad \text{if}\ \angle ADB = q,$$

therefore $\cos^2 p = \cot B\cot C$..............................(γ);

$$\cot B\tan p = \frac{\tan\frac{1}{2}c\sin c}{\sin q\sin p},\quad \cot C\tan p = \frac{\tan\frac{1}{2}b\sin b}{\sin q\sin p},$$

therefore $\dfrac{\cot B}{\cot C} = \dfrac{\sin^2\frac{1}{2}c}{\sin^2\frac{1}{2}b}$..................................(δ).

From (γ), (δ), $\cot^2 B = \cos^2 p\dfrac{\sin^2\frac{1}{2}c}{\sin^2\frac{1}{2}b}$; hence, remembering ($\beta$), we have

$$\cot B\sin\tfrac{1}{2}b = \sin\tfrac{1}{2}c\cos^{\frac{1}{2}}\tfrac{1}{2}(b+c)\cos^{\frac{1}{2}}\tfrac{1}{2}(b-c).$$

6574. (By W. B. Grove, B.A.)—Suppose a series of 73 cards to be painted with red, blue, yellow, and green, every card but one receiving at least one colour. Let it be observed that 21 have *some* part coloured red, 48 blue, 31 yellow, and 46 green: also 14 have both red and blue upon them, 16 both red and green, 14 both blue and yellow, 28 both blue and green, 20 both yellow and green, and 9 have all four colours. Also 15 are painted with blue *alone*, 6 with yellow alone, but none with either red or green alone. Find the laws (designed or accidental) according to which the colours are arranged. [This question is related to what Prof. Jevons calls the inverse logical problem, which Mr. Grove states that he has completely solved by Mr. McColl's logical notation.]

Solution by Dr. Macfarlane, F.R.S.E.

Let C denote the collection of cards, r red, b blue, y yellow, g green. Then the data are

$$C = 73,\quad C(1-r)(1-b)(1-y)(1-g) = 1,\quad Cr = 21,\quad Cb = 48,$$
$$Cy = 31,\quad Cg = 46,\quad Crb = 14,\quad Crg = 16,\quad Cby = 14,$$
$$Cbg = 28,\quad Cyg = 20,\quad Crbyg = 9,\quad Cb(1-r)(1-y)(1-g) = 15,$$
$$Cy(1-r)(1-b)(1-g) = 6,\quad Cr(1-b)(1-y)(1-g) = 0,$$
$$Cg(1-r)(1-b)(1-y) = 0.$$

From these equations I first obtain, by the elementary processes of the *Algebra of Logic*, the values of the other non-negative terms, namely, $Cry = 14$, $Crby = 14$, $Crbg = 9$, $Cryg = 9$, $Cbyg = 9$. By expanding the terms, I obtain the following laws:—

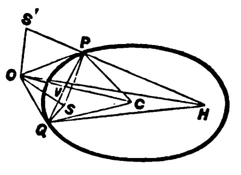
$$Crb(1-y) = 0,\quad Cry(1-b) = 0,\quad Cby(1-r) = 0\ \text{.........}\ (1, 2, 3),$$
$$Cr(1-g)(1-b)(1-y) = 0,\quad Cg(1-r)(1-b)(1-y) = 0\text{.........}(4, 5).$$

This set of laws evidently contains no redundancy.

The cards which are red and blue are yellow; those which are red and yellow are blue; those which are blue and yellow are red; those which are red, and not blue and not yellow, are green; those which are green, and not blue and not yellow, are red. [Or thus:—the cards that are red must be either both blue and yellow, or, if neither, green, and *vice versâ*.]

6171. (By the Rev. C. TAYLOR, D.D.)—Prove (1) that the trapezium bounded by a pair of tangents to an ellipse and the diameters to their points of contact is equal to the triangle whose sides are equal to the major axis and the focal distances of the point of concourse of the tangents; and (2) deduce therefrom solutions of Questions 2010 and 3099.

Solution by D. EDWARDES; H. HILLARY, M.A.; *and others.*



Let OP, OQ be the tangents, S, H, C the foci and centre. Then, evidently,

$$2\Delta PCQ = \Delta PSQ + \Delta PHQ;$$

hence $2\Delta OPQ + 2\Delta PCQ = OPSQ + OPHQ$.

Since the perpendiculars from O on SP, SQ, HP, HQ are equal, this gives $2OPCQ = p.AA'$, p being the length of these perpendiculars, and AA′ the major axis. If HP be produced to S′, so that $HS' = AA'$, then $OS = OS'$ and $\frac{1}{2}p.AA'$ = area of OHS′; hence area OPCQ = area OHS′. [From this theorem we have, in Quest. 2010 (*Reprint*, Vol. VII, p. 43) $AQ.AR \sin A + \ldots + \ldots = 2\Delta ABC$, &c.

Again, if CO meet PQ in V, we have, in Quest. 3099 (Vol. XIV, p. 74),

$$\Delta OHS' = OPQ + CPQ = OPQ\left(1 + \frac{CV}{VO}\right); \therefore \frac{OP.OQ}{OS'.OH} = \frac{\Delta OPQ}{\Delta OHS'} = \frac{OV}{OC}.]$$

6536. (By Professor BALL, F.R.S.)—If k be the constant term in the equation of a surface, and $\Delta = 0$ the condition necessary that this surface and three others pass through a point, what is the geometrical meaning of the roots of the equation $e^{-x\frac{d}{dk}}\Delta = 0$?

Solution by W. J. C. SHARP, M.A.

By the symbolical form of TAYLOR's Theorem, $e^{-x\frac{d}{dk}}\Delta(k) \equiv \Delta(k-x)$; and, if $U = k$ be the original surface, $U = k - x$ will be the equation to a surface which passes through the intersection of the other three, and is similar and similarly situated to $U = k$. Hence the roots determine these surfaces.

6430. (By J. W. SHARPE, M.A.)—If α, β, γ, δ, ρ be any five vectors, prove that the right part of the following expression vanishes,

$$\alpha\beta S\gamma\delta\rho + \beta\gamma S\alpha\delta\rho + \gamma\alpha S\beta\delta\rho + \gamma\delta S\alpha\beta\rho + \alpha\delta S\beta\gamma\rho + \beta\delta S\gamma\alpha\rho$$

Solution by the PROPOSER; S. RUGGERO; *and others.*

The right part of the expression is

$$V\alpha\beta S\gamma\delta\rho + V\gamma\delta S\alpha\beta\rho + V\alpha\left(-\gamma S\beta\delta\rho + \delta S\beta\gamma\rho\right) + V\beta\left(\gamma S\alpha\delta\rho + \delta S\gamma\alpha\rho\right)$$
$$= V\alpha\beta S\gamma\delta\rho + V\gamma\delta S\alpha\beta\rho + V\alpha\left(-\rho S\gamma\beta\delta + \beta S\delta\gamma\rho\right) + V\beta\left(\rho S\gamma\alpha\delta - \alpha S\delta\gamma\rho\right)$$
$$= -V\rho\left\{\alpha S\beta\gamma\delta - \beta S\gamma\delta\alpha\right\} - V\alpha\beta S\gamma\delta\rho + V\gamma\delta S\alpha\beta\rho$$
$$= -V\rho\left\{V\left(V\gamma\delta\,.\,V\beta\alpha\right)\right\} - V\alpha\beta S\gamma\delta\rho + V\gamma\delta S\alpha\beta\rho$$
$$= -V\left(V\mu\lambda - \rho\right) - \lambda S\mu\rho + \mu S\lambda\rho = -\left(\mu S\lambda\rho - \lambda S\mu\rho\right) - \lambda S\mu\rho + \mu S\lambda\rho = 0,$$

where λ and μ are put for $V\alpha\beta$, $V\gamma\delta$, respectively.

6519. (By W. S. McCAY, M.A.)—If two tangents be drawn to a nodal cubic from a point on the curve, prove that the line joining the points of contact envelopes a conic having triple contact with the cubic; and show that, if the conic be $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$, the conic is

$$5\left(x^2 + y^2 + z^2\right) + 22\left(yz + zx + xy\right) = 0.$$

Solution by W. H. BLYTHE, B.A.; Prof. NASH, M.A.; *and others.*

Taking θ^3, $(1-\theta)^3$, -1 to be the coordinates of any point on the cubic, (SALMON'S *Higher Curves*, p. 181), the equation of the tangent at (θ) is $(1-\theta)^2 x + \theta^2 y + \theta^2(1-\theta)^2 z = 0$; hence, if ϕ_1, ϕ_2 be the points of contact of tangents from θ, ϕ_1, ϕ_2 are roots of the equation $\phi^2 + 2\phi(\theta - 1) - \theta = 0$; therefore the equation of the line joining ϕ_1, ϕ_2 is

$$x\left(4\phi^2 - \phi + 1\right) + y\left(4\phi^2 - 7\phi + 4\right) + z\left(\phi^2 - \phi + 4\right) = 0;$$

and the envelop of this is

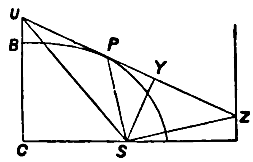
$$(x + 7y + z)^2 = 4\left(4x + 4y + z\right)\left(x + 4y + z\right),$$

or
$$5\left(x^2 + y^2 + z^2\right) + 22\left(yz + zx + xy\right) = 0.$$

This conic touches the cubic at the points $(-8, 1, 1)$, $(1, -8, 1)$, $(1, 1, -8)$, which are the points of contact of the tangents drawn to the cubic from the points of inflexion.

6665. (By Professor NASH, M.A.)—If two conics have the same focus S, and touch one another at P, prove that (1) the tangent at P will pass through Z, the intersection of the directrices corresponding to S; (2) a parabola described with focus S, touching the minor axes of these conics will touch PZ at Z, and its directrix will pass through P.

Solution by DOMENICO MONTESANO; G. HEPPEL, M.A.; *and others.*



Let CA, CB be the semi-axes, S the focus, of a conic, and let SY be perpendicular to the tangent at P which meets the directrix in Z and the minor axis in U. Then, because PSZ is a right angle, PZ . PY = SP².

Therefore, (1) if the positions of the tangent, point of contact, and focus are given, PZ is constant, and the directrices of all the conics pass through Z.

Also, (2) if a parabola touch PZ in Z, its directrix must pass through P, because ZSP is a right angle. And it is easy to show that the triangles CSU, PSY are similar, whence $\angle$ CUS = $\angle$ PSY = $\angle$ PZS, and therefore UC produced must be a tangent to the parabola. Hence, by the rule of identity, parabolas touching the minor axes of the conics must pass through Z. [This theorem is the reciprocal of the inverse of Euc. III. 11, 12, the centres of inversion and reciprocation being the same.]

6354. (By J. J. WALKER, M.A.)—If the coefficients of the binary quartic $(abcde)(xy)^4$ are connected by the relations $a^2d-3abc+2b^3=0$, $be^2-3cde+2d^3=0$, and if $ae-2bd$ does not vanish, prove that the quartic is a perfect square.

Solution by the PROPOSER.

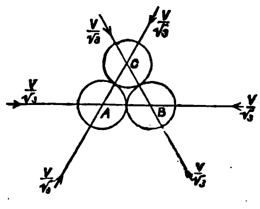
Multiplying the conditions by de, ab respectively, and subtracting, $a^2d^2e-2abd^3-ab^2e^2+2b^3de=0$, or $(ae-2bd)(ad^2-b^2e)=0$; consequently, if $ae-2bd$ does not vanish, $a:e=b^2:d^2$. Again, from the second condition, $\dfrac{3c}{e}=\dfrac{b}{d}+\dfrac{2d^2}{e^2}$. Hence $(a\ldots e)(xy)^4$ may be transformed into

$$e\left\{\frac{b^2}{d^2}x^4+4\frac{b}{d}x^3y+\left(2\frac{b}{d}+4\frac{d^2}{e^2}\right)x^2y^2+4\frac{d}{e}xy^3+y^4\right\},$$

which is equal to $$e\left(\frac{b}{d}x^2+2\frac{d}{e}xy+y^2\right)^2.$$

5978. (By D. EDWARDES.)—If 3 balls A, B, C, of equal mass and size, moving with the same velocity V in directions inclined at 120° to one another, impinge so that their centres form an equilateral triangle at the moment of impact; and if the coefficient of restitution between C and A or B be e, and between A and B be e'; show that A and B separate with a velocity $\frac{1}{3}(2e'+e)\,V\sqrt{3}$, it being assumed that compression ends at the same instant for all three balls.

Solution by Professor EVANS, M.A.; E. RUTTER; *and others.*



The V of A before impact is the resultant of two velocities $\frac{1}{3}V\sqrt{3}$ in directions AB, AC. If u, u' be the velocities of A, B after impact in AB, we have, by the third law of motion,

$$\tfrac{1}{3}mV\sqrt{3}-\tfrac{1}{3}mV\sqrt{3} = mu+mu',$$

therefore $\quad u' = -u;$

also $\quad u-u' = -\frac{1}{3}e'(V+V)\sqrt{3},$

or $\quad u = -\frac{1}{3}e'V\sqrt{3}.$

Similarly for the impact between A and C, if v be velocity of A after impact in AC; $v = -\frac{1}{3}eV\sqrt{3}$. The relative velocity of A to B after impact is evidently in direction AB, and the velocity of A in this direction

is $$-\tfrac{1}{3}e'V\sqrt{3}-\tfrac{1}{3}eV\sqrt{3}\cos 60^\circ = -\tfrac{1}{6}(2e'+e)V\sqrt{3}.$$

Hence relative velocity

$$= \tfrac{1}{6}\sqrt{3}\left[(2e'+e)V+(2e'+e)V\right] = \tfrac{1}{3}(2e'+e)V\sqrt{3}.$$

6559. (By R. RAWSON.) — If $E(x, y) = (x-y_1)\dots(x-y_n)$, and $\psi(x, y)$ be any function of (x, y), $f(x)$ being rational in x, and of lower dimension than n; show that (1) the integral

$$\int\left\{\frac{f(y_1)\,\psi(y_1\,.\,y)\,y_1^m}{\dfrac{dF(y_1\,.\,y)}{dy_1}}+\dots\frac{f(y_n)\,\psi(y_n\,.\,y)\,y_n^m}{\dfrac{dF(y_n\,.\,y)}{dy_n}}\right\}dy+C$$

is the coefficient of $\dfrac{1}{x^{m+1}}$ in $f(x)\displaystyle\int\frac{\psi(x, y)}{F(x, y)}\,dy$; and (2) show the application of this property to prove the celebrated theorem of Abel in ultra-elliptic functions.

Solution by the PROPOSER.

Since $F(x, y) = x-y_1\,.\,x-y_2\,.\,x-y_3\dots x-y_n$; then, by the usual theorem in the decomposition of rational fractions,

$$\frac{f(x)}{F(x, y)} = \Sigma\,.\,\frac{f(y_n)}{(x-y_n)\dfrac{dF(y_n\,.\,y)}{dy_n}} = \Sigma\,.\,\frac{f(y_n)}{\dfrac{dF(y_n\,.\,y)}{dy_n}}\left\{\frac{1}{x}+\frac{y_n}{x^2}+\frac{y_n^2}{x^3}+\dots\frac{y_n^m}{x^{m+1}}\right\}$$

$$= \frac{s}{x}+\frac{s_1}{x^2}+\dots\frac{s_m}{x^{m+1}}+\frac{s_{m+1}}{x^{m+2}}+\dots\frac{s_{2m}}{x^{2m+1}} \quad \dots\dots\dots\dots(1),$$

where $$s_m = \Sigma\,.\,\frac{f(y_n)\,y_n}{\dfrac{dF(y_n\,.\,y)}{dy_n}} \quad \dots\dots\dots\dots(2).$$

Multiply (1) successively by $a, a_1x, a_2x^2, a_3x^3, \ldots a_mx^m$, where a, a_1, a_2, &c. are functions of y; then

$$\left.\begin{aligned}
\frac{af(x)}{F(x,y)} &= \frac{as}{x} + \ldots \frac{as_m}{x^{m+1}} + \frac{as_{m+1}}{x^{m+2}} + \ldots \frac{as_{2m}}{x^{2m+1}} \\
\frac{a_1xf(x)}{F(x,y)} &= a_1s + \ldots \frac{a_1s_m}{x^m} + \frac{a_1s_{m+1}}{x^{m+1}} + \ldots \frac{a_1s_{2m}}{x^{2m}} \\
&\vdots \\
\frac{a_mx^mf(x)}{F(x,y)} &= a_msx^{m-1} + \ldots \frac{a_ms_m}{x} + \frac{a_ms_{m+1}}{x^2} + \ldots \frac{a_ms_{2m}}{x^{m+1}}
\end{aligned}\right\} \ldots\ldots(3).$$

From (3), we have

$$\text{coef. of } \frac{1}{x^{m+1}} \text{ in } (a + a_1x + \ldots a_mx^m)\frac{f(x)}{F(x,y)} = as_m + a_1s_{m+1} + \ldots a_ms_{2m} \ldots(4).$$

Restoring the values of s_m, s_{m+1}, &c., as given by (2),

$$\text{coefficient of } \frac{1}{x^{m+1}} \text{ in } (a + a_1x + \ldots a_mx^m)\frac{f(x)}{F(x,y)}$$

$$= \Sigma \cdot \frac{f(y_n)\,y_n^m}{\dfrac{dF(y_n \cdot y)}{dy_n}}(a + a_1y_n + \ldots a_my_n^m) \quad \ldots\ldots(5).$$

If, therefore, $\quad \psi(x, y) = a + a_1x + \ldots a_mx^m,$

we have $\quad \psi(y_n \cdot y) = a + a_1y_n + \ldots a_my_n^m.$

Substituting in (5),

$$\text{coefficient of } \frac{1}{x^{m+1}} \text{ in } \frac{f(x)\,\psi(x,y)}{F(x,y)} = \Sigma \cdot \frac{f(y_n)\,\psi(y_n \cdot y)\,y_n^m}{\dfrac{dF(y_n \cdot y)}{dy_n}} \ldots\ldots(6),$$

Integrating with respect to y,

$$\text{coef. of } \frac{1}{x^{m+1}} \text{ in } f(x)\int\frac{\psi(x,y)}{F(x,y)}\,dy = \Sigma \cdot \int\frac{f(y_n)\,\psi(y_n \cdot y)\,y_n^m}{\dfrac{dF(y_n \cdot y)}{dy_n}}\,dy + c \ldots(7).$$

Now, Abel has shown (pp. 282—299 of his "Math. Works," by Holmboe) that the sum of the (n) integrals

$$\pm\Sigma \cdot \int\frac{(fy_n)\,dy_n}{(y_n - a)[\phi(y_n)]^{\frac{1}{2}}} = \Sigma \cdot \int\frac{\lambda_1(y_n \cdot y)}{\dfrac{dF(y_n \cdot y)}{dy_n}}\,dy - \int\frac{\lambda(a,y)}{F(a,y)}\,dy + C \quad \ldots(8),$$

where $\quad \lambda(x,y) = 2f(x)\left\{\theta(x,y)\dfrac{d\theta_1(x,y)}{dy} - \theta_1(x,y)\dfrac{d\theta(x,y)}{dy}\right\},$

$$\lambda_1(x,y) = \frac{\lambda(x,y) - \lambda(a,y)}{x-a}; \quad \phi(x) = \phi_1(x)\,\phi_2(x),$$

$$[F(x,y) = \theta(x,y)^2\phi_1(x) - \theta_1(x,y)^2\phi_2(x).$$

By (7), $\quad \Sigma\displaystyle\int\frac{\lambda_1(y_n \cdot y)}{\dfrac{dF(y_n \cdot y)}{dy_n}}\,dy = \text{coefficient of } \frac{1}{x} \text{ in } \int\frac{\lambda_1(x,y)}{F(x,y)}\,dy$

$$= \text{coefficient of } \frac{1}{x} \text{ in } \int\frac{\lambda(x,y)}{(x-a)\,F(x,y)}\,dy.$$

Because the lowest negative power of x in $\dfrac{1}{(x-a)\,\mathrm{F}(x,y)}$ is $(n+1)$,

$$= \text{coefficient of } \frac{1}{x} \text{ in } \frac{2f(x)}{x-a}\int \frac{\theta(x,y)\dfrac{d\theta_1(x,y)}{dy} - \theta_1(x,y)\dfrac{d\theta(x,y)}{dy}}{\theta(x,y)^2\phi_1(x) - \theta_1(x,y)^2\phi_2(x)}\,dy$$

$$= \text{coef. of } \frac{1}{x} \text{ in } \frac{fx}{(x-a)\,[\phi(x)]^{\frac{1}{2}}}\log\frac{\theta(x,y)\,[\phi_1(x)]^{\frac{1}{2}} + \theta_1(x,y)[\phi_2(x)]^{\frac{1}{2}}}{\theta(x,y)\,[\phi_1(x)]^{\frac{1}{2}} - \theta_1(x,y)[\phi_2(x)]^{\frac{1}{2}}},$$

$$\int \frac{\lambda(a,y)}{\mathrm{F}(a,y)}\,dy = \int \frac{2f(a)\left\{\theta(a,y)\dfrac{d\theta_1(a,y)}{dy} - \theta_1(a,y)\dfrac{d\theta(a,y)}{dy}\right\}}{\theta(a,y)^2\phi_1(a) - \theta_1(a,y)^2\phi_2(a)}\,dy$$

$$= \frac{f(a)}{[\phi(a)]^{\frac{1}{2}}}\log\frac{\theta(a,y)\,[\phi_1(a)]^{\frac{1}{2}} + \theta_1(a,y)\,[\phi_2(a)]^{\frac{1}{2}}}{\theta(a,y)\,[\phi_1(a)]^{\frac{1}{2}} - \theta_1(a,y)\,[\phi_2(a)]^{\frac{1}{2}}},$$

which agree with ABEL's results.

6682. (By H. G. DAWSON.)—If $(n-1)$ of the roots of $(abc\ldots)(x,1)^n$ are connected by the relation $\Sigma(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta) = 0$, show that the remaining root w is given by $z = aw+b$,

$$z^3 - 3\frac{(n-1)^2}{n+1}\,.\,\mathrm{H}z + \frac{(n-2)(n-1)^3}{2(n+1)}\,.\,\mathrm{G} = 0,$$

where $\mathrm{H} = b^2 - ac$, $\mathrm{G} = 2b^3 - 3abc + a^2$; and hence deduce Quest. 6638.

Solution by Rev. T. R. TERRY, M.A.; CHRISTINE LADD; *and others.*

If $ax+b = y^n$, we have

$$y^n - \tfrac{1}{2}n(n-1)\,\mathrm{H}y^{n-2} + \tfrac{1}{3}n(n-1)(n-2)\,\mathrm{G}y^{n-3} + \ldots = 0 \;\ldots\ldots\ldots(1),$$

where $n-1$ roots are connected by the relation

$$\Sigma(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta) = 0,$$

which is equivalent to

$$(n-2)(n-3)\,\Sigma\alpha^3 - 3(n-3)\,\Sigma\alpha^2\beta + 12\Sigma\alpha\beta\gamma = 0 \ldots\ldots\ldots\ldots\ldots(2).$$

But if z be the remaining root of (1), we have obviously

$$\Sigma\alpha = -z,\quad \Sigma\alpha\beta = z^2 - \tfrac{1}{2}n(n-1)\,\mathrm{H},$$

$$\Sigma\alpha\beta\gamma = -z^3 + \tfrac{1}{2}n(n-1)\,\mathrm{H}z - \tfrac{1}{3}n(n-1)(n-2)\,\mathrm{G};$$

whence $\qquad \Sigma\alpha^2\beta = 2z^3 - n(n-1)\,\mathrm{H}z + \tfrac{1}{2}n(n-1)(n-2)\,\mathrm{G},$

and $\qquad \Sigma\alpha^3 = -z^3 - \tfrac{1}{2}n(n-1)(n-2)\,\mathrm{G};$

therefore, substituting in (2),

$$z^3 - 3\frac{(n-1)^2}{n+1}\,.\,\mathrm{H}z + \frac{(n-2)(n-1)^3}{2(n+1)}\,.\,\mathrm{G} = 0.$$

6650. (By D. EDWARDES.)—If tangents are drawn to the circumscribed circle of a triangle ABC at the angular points, and if Δ be the area of the triangle so formed, and Δ_1 that of the orthocentric triangle of ABC, prove, with the usual notation, that $\Delta^{\frac{1}{2}} : \Delta_1^{\frac{1}{2}} = \mathrm{R} : \rho$.

Solution by G. EASTWOOD, M.A, ; J. O'REGAN ; *and others.*

It is well known that the angles of the orthocentric triangle are $\pi-2\mathrm{A}$, $\pi-2\mathrm{B}$, $\pi-2\mathrm{C}$, and it is obvious from a figure that the angles of the triangle formed by the tangents at A, B, C are equal to the same three quantities. Therefore these two triangles are similar, and therefore their areas are in the duplicate ratio of their inscribed circles.

6611. (By the Rev. T. R. TERRY, M.A. Suggested by Question 6590.) —If there be any distribution of mass M in space, and I_a, I_b, I_c be its moments of inertia with regard to any three parallel lines A, B, C, and if a, b, c be the distances between the lines B and C, C and A, A and B, respectively; prove that the moment of inertia about a parallel line through the centre of gravity of the mass is

$$\frac{a^2\mathrm{I}_a^2+\ldots-(a^2+b^2-c^2)\,\mathrm{I}_a\mathrm{I}_b-\ldots-a^2\,(b^2+c^2-a^2)\,\mathrm{MI}_a-\ldots+\mathrm{M}^2a^2b^2c^2}{\mathrm{M}\,(a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2)}.$$

Solution by G. F. WALKER, M.A. ; A. ANDERSON, B.A. ; *and others.*

Let M be the mass, G the centre of gravity, and K the moment of inertia about it ; $\mathrm{I}_a = \mathrm{K}+\mathrm{M}\,.\,\mathrm{AP}^2$, $\mathrm{I}_b = \mathrm{K}+\mathrm{M}\,.\,\mathrm{BP}^2$,

$$\mathrm{I}_c = \mathrm{K}+\mathrm{M}\,.\,\mathrm{CP}^2.$$

Since A, B, C, G are four points in a plane, we have the following relation between their distances (d, e, f denote AG, BG, CG, respectively)—

$$a^2\,(d^2-e^2)\,(d^2-f^2)+b^2\,(e^2-f^2)\,(e^2-d^2)+c^2\,(f^2-d^2)\,(f^2-e^2)$$
$$+\,a^2d^2\,(a^2-b^2-c^2)+b^2e^2\,(b^2-c^2-a^2)+e^2f^2\,(c^2-a^2-b^2)+a^2b^2c=0$$

(SALMON, p. 34). Whence (multiplying by M^2 and substituting),

$$a^2\,(\mathrm{I}_a-\mathrm{I}_b)\,(\mathrm{I}_a-\mathrm{I}_c)+\ldots-a^2\,(b^2+c^2-a^2)\,\mathrm{M}\,(\mathrm{I}_a-\mathrm{K})-\ldots+\mathrm{M}^2a^2b^2c^2=0,$$

or $$\mathrm{K}=\frac{a^2\mathrm{I}_a^2+\ldots-(b^2+c^2-a^2)\,\mathrm{I}_b\mathrm{I}_c-\ldots-a^2\,(b^2+c^2-a^2)\,\mathrm{MI}_a-\ldots+\mathrm{M}^2a^2b^2c^2}{\mathrm{M}\,(a^4+\ldots-2b^2c^2-\ldots)}.$$

6201. (By CHRISTINE LADD.)—AB is the vertical diameter of a circle ; ball descends down the chord AC, and, being reflected by the plane BC, escribes its path as a projectile ; find the average range of the ball on e diameter CD, supposing all coefficients of friction relative to the descent of the ball on the chord to exist for which motion is possible.

Solution by W. J. C. Sharp, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

If AC make an angle α with AB, μ be the coefficient of friction, and r the radius of circle; we have

$$v \text{ (the velocity of impact)} = 2\left\{rg\cos\alpha\,(\cos\alpha-\mu\sin\alpha\right\}^{\frac{1}{2}},$$

and the velocity of reflection $= ev$, where e is the elasticity; the two equations $0 = ev\,.\,t-\frac{1}{2}g\sin 2\alpha\,.\,t^2$ and $s = \frac{1}{2}g\cos 2\alpha t^2$ determine the range s.

Therefore $$s = \frac{2e^2v^2\cos 2\alpha}{g\sin^2 2\alpha} = 8r\,\frac{e^2\cos 2\alpha\cos\alpha}{\sin^2 2\alpha}\left\{\cos\alpha-\mu\sin\alpha\right\}$$

therefore the average from $\mu=0$ to $\mu=1$

$$= \int_0^1 s\,d\mu = gr\,\frac{e^2\cos 2\alpha\cos\alpha}{\sin^2 2\alpha}\left\{\cos\alpha-\tfrac{1}{2}\sin\alpha\right\}$$
$$= e^2r\,(\cot^2\alpha-1)\,(2-\tan\alpha).$$

[It is assumed that both α and e are constants; if these are to be considered as variable, the results may be obtained from those given above by further integration.]

6492. (By J. Hammond, M.A.)—Prove that the sum of the infinite series $\dfrac{1}{m}+\dfrac{1}{m(m+1)}+\dfrac{1}{m(m+1)(m+2)}+\ldots = \displaystyle\int_0^1 e^x(1-x)^{m-1}\,dx.$

Solution by W. J. C. Sharp, M.A.; A. McMurchy, B.A.; *and others.*

$$\frac{x^m}{m}+\frac{x^{m+1}}{m(m+1)}+\frac{x^{m+2}}{m(m+1)(m+2)}+\&c.$$
$$=\left\{\left(\frac{d}{dx}\right)^{-1}+\left(\frac{d}{dx}\right)^{-2}+\left(\frac{d}{dx}\right)^{-3}+\&c.\right\}x^{m-1}$$
$$=\left(\frac{d}{dx}-1\right)^{-1}x^{m-1} = e^x\int_0^1 e^{-x}x^{m-1}\,dx;$$

therefore $$\frac{1}{m}+\frac{1}{m(m+1)}+\frac{1}{m(m+1)(m+2)}+\&c.$$
$$= e\int_0^1 e^{-x}x^{m-1}\,dx = \int_0^1 e^y(1-y)^{m-1}\,dy \quad \text{if } y = 1-x.$$

6287. (By the Editor.)—The base AB of a triangle ABC being given in position and magnitude, and the side AC in magnitude only; trace the locus of the centre of the inscribed circle (1) generally, and (2) when AB = AC; also (3) find when the inscribed circle is a maximum, showing that then its radius is twice the distance of the third side of the triangle from the centre of the circumscribed circle; and state (4) what are the loci of the *escribed* circles.

I. *Solution by* W. J. C. Sharp, M.A.

1. Take the given side AB as axis of x, and A as origin of rectangular coordinates, and put AC $= b$, AB $= c$, and (x, y) for the centre of the inscribed circle; then we have

$$\tan\tfrac{1}{2}A = \frac{y}{x},\quad \tan\tfrac{1}{2}B = \frac{y}{c-x},\quad \tan\tfrac{1}{2}C = \frac{y}{b-x},$$

and $\tan\frac{1}{2}A\tan\frac{1}{2}B + \tan\frac{1}{2}B\tan\frac{1}{2}C + \tan\frac{1}{2}C\tan\frac{1}{2}A = 1$;

hence $y^2(b+c-x) = x(b-x)(c-x)$ or $(x^2+y^2)(b+c-x) = bcx$ ......(1)

is the equation to the locus, which is therefore a circular cubic through the origin, having the line $x = b+c$ for its asymptote.

2. When $b = c$ the equation becomes

$$(x^2+y^2)(x-2c)+c^2x = 0 \text{ or } x(x-c)^2+y^2(x-2c) = 0 \quad\text{.........(2)},$$

which has a double point at $(c, 0)$, the tangents whereat are

$$c(x-c)^2-cy^2 = 0, \text{ that is, } x+y-c = 0 \text{ and } x-y-c = 0.$$

Equation (2) may be written $(x^2+y^2)(x-c)^2 = c^2y^2$ ....(3);

hence the curve is such that, if P be any point on it,

$$AB\tan PBX = AP.$$

3. If the circle be a maximum, so is y^2, and the values of x are given by

$$2x^3-4(b+c)x^2+2(b+c)^2x-bc(b+c) = 0 \quad\text{...............(4)},$$

and if $b=c$ by $(x-c)(x^2-3cx+c^2) = 0$ ....(5).

The root $x = c$, of (5), gives y a minimum; and the other roots $x = \frac{1}{2}(3\pm\sqrt{5})c$ give $y^2 = \dfrac{7\pm3\sqrt{5}}{1\mp\sqrt{5}}c^2 = \frac{1}{4}(\mp5\sqrt{5}-11)c^2$, so that $\frac{1}{2}(3+\sqrt{5})c$ makes y impossible, and $x = \frac{1}{2}(3-\sqrt{5})c$ makes y a maximum.

4. For the circle touching AB and the other sides produced, we have

$$\tan\tfrac{1}{2}A = -\frac{x}{y},\quad \tan\tfrac{1}{2}B = -\frac{c-x}{y},\quad \tan\tfrac{1}{2}C = -\frac{y}{b+x};$$

and the equation is $(x^2+y^2)(x+b-c) = bcx$...............................(6),

which is also a circular cubic through the origin, whose real asymptote is $x+b-c = 0$, the satellite of infinity being, as before, $x = 0$.

If $b = c$, the curve degenerates into a circle whose centre is A and radius c, and the axis of y.

The locus of the circle touching AC and the other sides produced is the similar curve $(x^2+y^2)(x+c-b) = bcx$ ....(7),

which becomes identical with (6) if $b = c$.

The equation to the locus of the centre of the circle escribed to the side BC is $(x^2+y^2)\{x-b-c\}+bcx = 0$ ....(8),

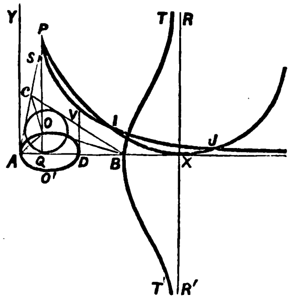
which is the same curve as the locus (1) of the centre of inscribed circle.

5. It is evident that, in each of the loci, the origin is the double focus; that, in (2), $(c, 0)$ is a focus; that, in (1), (6), (7), the tangents at the points where $y = 0$ cuts the curves are, respectively,

$$(x = 0,\ x-b = 0,\ x-c+0),\quad (x = 0,\ x+b = 0,\ x-c = 0),$$
$$(x = 0,\ -b = 0,\ x+c = 0);$$

and that the corresponding satellites in (1), (6), (7) are $x-b-c = 0$, $x+b-c = 0$, $x-b+c = 0$, that is to say, in each case, the asymptote.

II. *Solution by the* PROPOSER.



In the triangle ABC, let O be the centre of the inscribed circle, and OQ ($=y$) its radius; then, if AC = AD = b, AB = c, BC = z, AX = $b+c=e$, AO = r, $\angle$ OAB = θ, and AQ = x, so that (x, y) and (r, θ) are the rectangular and polar coordinates of O, we have

$$\tfrac{1}{2}(e-z) = x = y\cot\theta \;\ldots\ldots(1),$$

$$\tfrac{1}{2}(e+z) = \frac{bc}{2y}\sin 2\theta = \frac{bc}{x}\cos^2\theta \;..(2);$$

whence, eliminating z, θ, we find, for the locus of O, any one of the following equations :—

$$(e-x)\,r^2 = bcx, \quad (e-x)\,y^2 = x\,(x-b)\,(x-c) \;\ldots\ldots\ldots\ldots(3, 4),$$

$$(r^2+bc)\cos\theta = er, \quad \frac{dy}{dx} = \frac{\tfrac{1}{2}bce - x\,(e-x)^2}{y\,(e-x)^2} \;\ldots\ldots\ldots\ldots(5, 6).$$

This cubic locus consists of a loop AODO′, and (in general) two infinite branches TBT′ along the perpendicular asymptote RXR′. When $b=c$, the curve is continuous at B, where the two branches cross each other. Differentiating, in (2), with respect to θ, we find

$$\frac{2bc}{y}\cos 2\theta - \frac{bc}{y^2}\sin 2\theta\,\frac{dy}{d\theta} = \frac{dz}{d\theta} = \frac{2bc}{z}\sin 2\theta,$$

the second equality resulting from differentiating

$$(1)\times(2) \equiv e^2 - z^2 = 2bc\,(1+\cos 2\theta);$$

whence we obtain
$$\frac{dy}{d\theta} = \frac{2y}{z}\,(z\cot 2\theta - y) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7);$$

hence, when the inscribed circle is a maximum, its radius

$$y = z\cot 2\theta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

which proves part (3) of the Question.

From (1), (2), (8), we have $x = (e-x)\cos 2\theta$, $2x\,(e-x)^2 = bce$...(9, 10). Equation (10) determines the maximum radius of the inscribed circle, agreeing with Mr. SHARP's equation (4), and with what, in another way, is obtainable from (6) above. From (10) we have, in the figure,

$$\text{AB}\,.\,\text{AD}\,.\,\text{AX} = 2\text{AQ}\,.\,\text{QX}^2 \;\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

Hence, to determine the triangle that has the maximum inscribed circle, we have the following construction :—

With AX as parameter and XR as axis draw the parabola PIXJ; and through the point V (where DV is perpendicular to AX and = $\tfrac{1}{2}$AB) along the asymptotes AX, AY, draw the rectangular hyperbola PVIJ; from P draw POQ perpendicular to AX; and construct the triangle ABC so as to have the two given sides AB, AC, and the third side BC = AX − 2AQ; then the circle inscribed in this triangle will be a maximum.

For, from the parabola and the hyperbola respectively, we have

$$\text{AX}\,.\,\text{PQ} = \text{QX}^2,\ \text{AD}\,.\,\text{DV} = \text{AQ}\,.\,\text{QP} \text{ or } \text{AD}\,.\,\text{AB} = 2\text{AQ}\,.\,\text{PQ};$$

hence AB . AD . AX = 2AQ . QX², which agrees with (11).

After determining P, and Q therefrom, as above, the triangle may be

otherwise constructed, by the aid of equation (9), by inflecting from A to PQ a line AS = QX, taking thereon AC = b, and joining BC.

Writing (10) in the form $F(x) \equiv 2x(e-x)^2 - bce = 0$, we find that, when $x = 0, b, c, e, \frac{1}{2}e$, then $F(x) = -bce, (c-b)bc, -(c-b)bc, -bce, \frac{1}{27}e\{8(c-b)^2 + 5bc\}$; hence (since values substituted for x in an equation $F(x) = 0$ give *like* signs when they comprise between them an *even* number of roots, and *unlike* signs when they comprise an *odd* number) the roots of the cubic (10), that is, of $F(x) = 0$, are in the intervals $(0, b)$, (b, c), $(e, \frac{1}{2}e)$, therein agreeing with what is shown in the above graphical construction.

From (8) we have $y \tan A = s = y(\cot \frac{1}{2}B + \cot \frac{1}{2}C)$; hence, in the triangle that has the maximum inscribed circle, there are the following relations among the angles:—

$$\tan A = \cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C, \quad \cos A = 2 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C,$$

$$1 + \cos A = \cos B + \cos C, \quad \cos B + \cos C - \cos A = 1.$$

The infinite branch TBT′ of the cubic locus is the locus of the circle escribed to the side BC, and the locus of the inscribed circle is the loop AODO′.

6745. (By A. McIntosh, B.A.)—If in a nodal cubic, with the nodal tangents at right angles, a right-angled triangle be inscribed having the right angle at the double point; show that (1) the hypotenuse passes through a fixed point on the curve; and (2) in the case of the Folium of Descartes, this point is the vertex or apse of the loop.

Solution by Professor Townsend, F.R.S.

The equation of the curve, to the nodal tangents as axes of coordinates, being $u_3 + xy = 0$, where $u_3 = ax^3 + by^3 + 3hx^2y + 3ky^2x$, that of the three lines from the origin to its three intersections with any arbitrary line $lx + my = 1$ is consequently $u_3 + (lx + my)xy = 0$, which, when two of them are (as by hypothesis) at right angles to each other, must be of the form $(px + qy)(x^2 - y^2 + 2rxy) = 0$; hence we must have $p = a$ and $q = -b$, which shows that the third line is fixed, and therefore &c., as regards the general property in question, from which the particular follows immediately from the consideration that, where the curve has an axis of figure, the point must be similarly related to its two symmetrical halves, and be therefore its apsidal point on the axis.

A nodal cubic reciprocating, to any circle having its centre at the node, into a tricuspidal quartic having a double tangent at infinity; and inverting, to the same, into a unicircular quartic having a triple point at the node; we infer consequently from the above, by application of the latter and former processes respectively, that—

(a) A quartic, having a triple point, being supposed to intersect infinity at the two circular points, and also at a real pair of rectangular points; the circle passing through the triple point, and through the extremities of a variable chord subtending it at right angles, passes through a second fixed point on the curve.

(*b*) A quartic, having a triad of cuspidal points, being supposed to have double contact with infinity at a pair of rectangular points; every tangent to either branch into which it is divided by its bitangent chord intersects with the orthogonal tangent to the opposite branch on a fixed tangent to the curve.

Every right angle, whatever be its position, being divided harmonically by the connectors of its vertex with the two circular points at infinity in its plane; we infer again, consequently, from the above, by projection in its most general sense, that—

(*c*) A variable chord of a nodal cubic, divided harmonically in every position by any fixed pair of nodal chords harmonically conjugate to each other with respect to the nodal tangents, passes in every position through a fixed point on the curve; that, viz., at which (see 4995) the tangents at the opposite extremities of the two nodal chords intersect on the curve.

6373. (By Professor SYLVESTER, F.R.S., D.C.L.)—If

$$fx = Ax^n + Bx^{n-1} + Cx^{n-2} + Dx^{n-3} \ldots + L$$

$$= ax^n + nbx^{n-1} + n\,.\,\tfrac{1}{2}(n-1)\,cx^{n-2} + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\,dx^{n-3} \ldots + l,$$

prove that (1) fx cannot have more real roots than there are continuations of sign in the series $A^2 : B^2 - AC : C^2 - BD : \ldots : L^2$; (2) fx cannot have more real roots than there are continuations of sign in the series

$$b^2 : c^2 - \tfrac{3}{4}bd : d^2 - \tfrac{8}{9}ce : e^2 - \tfrac{24}{25}df : f^2 - \tfrac{35}{36}eg : \ldots : l^2.$$

Solution by the PROPOSER.

The above conclusions follow from putting, in the formula $\gamma_r = \dfrac{C-1+r}{C+r}$, in TODHUNTER's chapter "On NEWTON's Rule and SYLVESTER's Theorem" in his "Theory of Equations" (p. 249), for (1) $C = \infty$, for (2) $C = 0$.

Observe that C is limited to be *anything* not *between* 0 and $-n$, or, as we may phrase it, anywhere on the arc of quantities from 0 to $-n$, passing through $\pm\infty$.

NEWTON's Rule corresponds to putting C equal to *one* extremity of this circle, viz. $-n$. Theorem (1) corresponds to putting C = *to the half-way quantity*, viz. $\pm\infty$. Theorem (2) corresponds to putting C = to the *other* extremity, viz. 0.

As an example of (1), take $x^n + x^{n-1} + \ldots + n + 1 = 0$. NEWTON's rule only indicates the existence of some imaginary roots; then, considering 0 as $\pm$ at will, (1) proves that all the roots, or all but one, are imaginary.

Nobody seems to have noticed that, when we take $C = -n$ (NEWTON's rule case), the quadratic quantities $G_1, G_2, \ldots G_{n-1}$ intermediate between the first and last, which are merely the pillar-post signs + +, instead of being of the dimensions

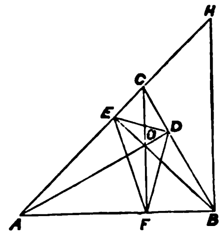
$$2(n-1),\ 2(n-2),\ \ldots,\ 2\,.\,2,\ 2\,.\,1\ \text{(each)},$$

lose two dimensions, becoming of the respective degrees

$G_{n-1},\ G_{n-2},\ G_2,\ G_1,\ 2(n-2),\ 2(n-3),\ \ldots,\ 2,\ 0,\quad G_1$ being a constant.

5546. (By the Editor.)—In a triangle ABH, right-angled at B and having AB = BH, take HC = $\frac{1}{3}$HA, and join BC; and let O be the orthocentre of the triangle ABC, DEF its orthocentric triangle, and a', b', c', ρ, Δ' the sides and inscribed radius of this orthocentric triangle, the usual notation referring to the parts of the triangle ABC; then prove that (1) $\tan A = 1$, $\tan B = 2$, $\tan C = 3$; (2) $\Delta = 5\Delta' = b^2 - a^2 = \frac{1}{3}c^2$; (3) AF = 2FB, BD = $\frac{3}{2}$DC, CE = $\frac{1}{3}$EA; (4) AO = 5OD, BO = 2OE, CO = OF; (5) $\rho = \frac{1}{5}R$; and (6) develop other properties of the system.

Solution by G. Heppel, M.A.; J. O'Regan; *and others.*



In Booth's *New Geometrical Methods* (Vol. II., p. 301) it is shown that

$$\frac{\Delta'}{\Delta} = \frac{\rho}{R} = 2\cos A\cos B\cos C \;\ldots\ldots\ldots (1).$$

We have also AD = CD tan C, OD = CD cot B,

therefore $\dfrac{AD}{OD} = \tan B \tan C$;

$$\left.\begin{aligned} \text{or}\qquad AO &= OD\,(\tan B\tan C - 1),\\ \text{similarly}\quad BO &= OE\,(\tan C\tan A - 1),\\ CO &= OF\,(\tan A\tan B - 1),\end{aligned}\right\}\;\ldots\ldots\ldots (2).$$

And obviously AF : FB = tan B : tan A,

BD : DC = tan C : tan B, CE : EA = tan A : tan C ....(3).

Now, from the figure, AF = CF = 2BF, *i.e.*, tan A = 1, tan B = 2, and tan C = $-\tan(A+B) = 3$.

Substituting for A, B, C in (1), (2), (3), we obtain

$$\Delta = 5\Delta',\;\; \rho = \tfrac{1}{5}R,\;\; AO = 5\,.\,OD,\;\; BO = 2\,.\,OE,\;\; CO = OF,\;\; AF = 2FB,$$
$$BD = \tfrac{3}{2}DC,\quad CE = \tfrac{1}{3}EA.$$

Again, $\Delta = \dfrac{CF}{HB}\,.\,ABH = \tfrac{2}{3}ABH = \tfrac{1}{3}c^2$, and $b^2 - a^2 = AF^2 - BF^2 = \tfrac{1}{3}c^2$.

6571 & 6598. (By C. B. S. Cavallin, M.A.)—Taking the variation of the Croftonian integral $\iint (\theta - \sin\theta)\, dx\, dy = \frac{1}{2}L^2 - \pi\Omega$, (*Phil. Trans.*, 1868, p. 188), on the supposition that the contour of reference (of length L inclosing an area Ω) changes to another nearly situated, generated by prolonging each radius ρ of curvature in Ω a length $\mu f(\rho)$, where μ is an infinitely small constant, prove that we obtain

$$\iint \left\{ \frac{f(\rho_1)}{t_1} + \frac{f(\rho_2)}{t_2} \right\} \sin^2 \tfrac{1}{2}\theta\, dx\, dy = \tfrac{1}{2}L \int^{2\pi} f(\rho)\, d\phi,$$

where t_1, t_2 are the lengths of the tangents, drawn from the point (x, y) to Ω; ρ_1, ρ_2 the radii of curvature at the corresponding points of contact; ρ the radius of curvature at an arbitrary point of the curve; and ϕ the inclination of this radius to a fixed line in the plane; the integration extending over the whole plane outside Ω.

(6598.) Prove that, (1) by starting from the Croftonian integral

$$\iint \theta\, dx\, dy = \pi\Theta - \int_0^{2\pi} \Sigma\, d\omega, \quad (ib., \text{p. } 190)$$

and only varying the *inner* contour, we get

$$\iint \left\{ \frac{f(\rho_1)}{t_1} + \frac{f(\rho_2)}{t_2} \right\} dx\, dy = \int_0^{2\pi} cf(\rho)\, d\omega,$$

where θ is the angle which at any point within the annulus (Θ) between an interior and exterior contour is subtended by the former; c is a chord of the latter intercepting a segment Σ of it and touching the former; ω the angle which c makes with a fixed line, and the other notation as in Quest. 6571. Also, (2) by only varying the *outer* contour, prove that we get

$$\int_0^{2\pi} (\pi - \theta) \rho f(\rho)\, d\phi = \int_0^{2\pi} \left\{ \int_0^{\beta} \rho f(\rho)\, d\phi \right\} d\omega,$$

where ρ is now the radius of curvature of the outer contour at the point at which the inner subtends an angle θ; β the angle between the normals at the ends of c, and the rest of the notation as in (6571), but the integration extends only over the annulus between the two contours.

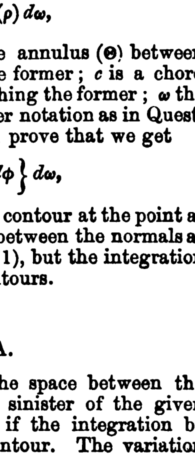
Solution by J. J. WALKER, M.A.

Calling the area of the new contour Ω', since in the space between the two contours $\theta - \sin\theta$ has the constant value π, the sinister of the given equality must be increased by the term $\pi(\Omega' - \Omega)$ if the integration be extended only over the space exterior to the new contour. The variation thus gives (θ', L$'$ referring to the new contour)

$$\iint (\theta' - \theta - \sin\theta' + \sin\theta)\, dx\, dy = \text{L}(\text{L}' - \text{L}),$$

But the figure shows at once that

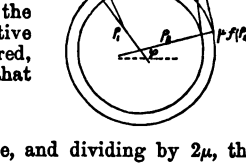
$$\theta' - \theta = \mu \left(\frac{f(\rho_1)}{t_1} + \frac{f(\rho_2)}{t_2} \right),$$



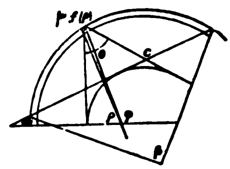
and $\sin\theta' - \sin\theta = (\theta' - \theta)\cos\theta$. Again, if the elements of L$'$ and L limited by two consecutive normals, inclined at an angle $d\phi$, be considered, their difference is plainly equal to $\mu f(\rho)\, d\phi$, so that

$$\text{L}' - \text{L} = \mu \int_0^{2\pi} f(\rho)\, d\phi.$$

Substituting these values in the equality above, and dividing by 2μ, the first result is proved.



For the second result, the sinister of the equality from which we are to start must be increased by the term $\pi(\Theta - \Theta')$, if the integration is to extend only over the annulus between the new interior contour and the exterior contour, since within the two interior contours θ has the constant value π; so that the variation gives $\iint (\theta' - \theta)\, dx\, dy = \int_0^{2\pi} (\Sigma - \Sigma')\, d\omega$. But, as above, $\theta' - \theta = \mu(\ldots)$ and $\Sigma - \Sigma'$ plainly is equal to $\mu cf(\rho)$. Substituting and dividing by μ, the second result is obtained.



For the third result, the variation of the sinister in this case—since at corresponding points on the new and given exterior contours θ may be regarded as having the same values, retaining only terms multiplied by the first power of μ—will be $\iint \theta \, dx \, dy$, taken only over the space between those contours, which is plainly equal to $\mu \int_0^{2\pi} \theta \rho f(\rho) \, d\phi$. For the variation of the dexter $\pi(\Theta' - \Theta) = \pi$ times the same integral; and $\Sigma' - \Sigma$ (which is the portion cut off by c of the elementary annulus contained between the new and given exterior contours) is similarly equal to $\mu \int_0^{\beta} \rho f(\rho) \, d\phi$. Substituting these values, transposing the term multiplied by π, and dividing by $-\mu$, the final result, as given in Quest. 6598, is obtained.

[Mr. CAVALLIN is to be congratulated on having obtained these new integral relations in so ingenious a manner from Prof. CROFTON's integrals.]

6651. (By E. W. SYMONS, B.A.)—A parabola has contact of the 4th order with a given circle; prove that its focus moves on one concentric circle, and that its directrix touches another.

Solution by the Rev. T. R. TERRY, M.A., F.R.A.S.; E. RUTTER; *and others.*

Since the contact is of the 4th order, the point of contact must be at the vertex of the parabola. But radius of curvature at the point $= 2SA$. Hence, if R be the radius of the given circle, the locus of the focus is a concentric circle of radius $\frac{1}{2}$R, and envelope of directrix is a concentric circle of radius $\frac{3}{2}$R.

6673. (By the Rev. W. A. WHITWORTH, M.A.)—Find a triangle whose altitude and sides are expressed by four consecutive integers, and show that only one such triangle exists.

Solution by the PROPOSER.

If we first tabulate all the triangles whose sides are expressed by consecutive integers and whose areas are rational, we find

3, 4, 5 (area = 6); 13, 14, 15 (area = 84); 51, 52, 53 (area = 1170); &c.

The second of these (13, 14, 15) has an altitude 12, and thus fulfils the conditions of the question. And it is plain that no triangle whose sides are *higher* consecutive integers can possibly fulfil the conditions. For, as we proceed to higher integers, such series of triangles will approximate

more and more to an *equilateral* triangle, and the ratio of the perpendicular to any of the sides must approximate to the ratio $\sqrt{3} : 2$, or approximately 6 : 7. But the ratio of an integer to any one of the next three consecutive integers recedes from this value when the integer is higher than 18. Therefore it is impossible that the conditions of the question can be fulfilled by any higher numbers. Hence the only triangle is the triangle whose altitude is 12, and its sides 13, 14, 15.

6427. (By R. A. Roberts, M.A.)—Show that the cubics whose equations in rectangular coordinates are

$$(a-b)\,xy^2 + g\,(y^2-x^2) - 2fxy - k^2\,(ax+g) = 0,$$
$$(a-b)\,x^2y + f\,(y^2-x^2) + 2gxy + k^2\,(by+f) = 0,$$

cut each other at right angles at their seven finite points of intersection.

Solution by G. F. Walker, M.A.; Prof. Matz, M.A.; *and others.*

We have, from the cubics,

$$[2xy\,(a-b) + 2gy - 2fx]\,\frac{dy}{dx} + (a-b)\,y^2 - 2gx - 2fy - k^2a = 0,$$

and $\quad [(a-b)\,x^2 + 2fy + 2gx + k^2b]\,\dfrac{dy}{dx} + 2xy\,(a-b) + 2gy - 2fx = 0;$

and, if they are at right angles, we must have $xy\,(a-b) + gy - fx = 0$, or $(a-b)x^2 + 2fy + 2gx + k^2b + (a-b)y^2 - 2gx - 2fy - k^2a = 0$, *i.e.*, $(x^2+y^2) = k^2$.

Now, if we multiply the first equation by y, and the second by x, and add, we get

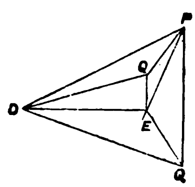
$$(x^2+y^2)\,xy\,(a-b) + (gy+fx)\,(y^2-x^2) + 2xy\,(gx-fy) - k^2xy\,(a-b) - k^2\,(gy-fx) = 0,$$

which is $\quad (x^2+y^2-k^2)\,[xy\,(a-b) + gy - fx] = 0;$

and therefore the curves cut at right angles.

6570. (By Prof. Genese, M.A.)—The angle which a fixed diameter of a rectangular hyperbola subtends at a variable point is divided into two parts, whose sines are in a given ratio; prove that the dividing line passes through a fixed point.

Solution by A. McMurchy, B.A.; Christine Ladd; *and others.*



Let the dividing line meet the curve again at Q, and let DE be the fixed diameter, then

$$\angle\,QDP = QEP$$

or supplement; hence diameter of circle PQD

$= PQ \operatorname{cosec} QDP =$ diameter of circle QED;

$\therefore\;$ QD : QE = sin QPD : sin QPE = constant,

and Q is a fixed point. There will be another fixed point corresponding to the lines dividing the angle DPF externally.

6725. (By W. H. H. HUDSON, M.A.)—If Δ be the area, and P the perimeter, of a triangle drawn on a sphere of radius r, and if Δ', P′ be the area and perimeter of the polar triangle; prove (1) that

$$\frac{\Delta}{r^2}+\frac{P'}{r}=\frac{\Delta'}{r^2}+\frac{P}{r}=2\pi;$$

and (2) state the corresponding proposition for any spherical polygon.

Solution by R. TUCKER, M.A.; CHRISTINE LADD; *and others.*

$$\frac{\Delta}{r^2}=A+B+C-\pi,\quad P'=a'+b'+c';$$

therefore $\frac{\Delta}{r^2}+P'=(A+a')+(B+b')+(C+c')-\pi=2\pi=\frac{\Delta'}{r^2}+P.$

(2) $$\frac{S}{r^2}=\Sigma-(n-2)\pi,\quad P'=a'+b'+c'+\dots;$$

$$\therefore\ \frac{S}{r^2}+P'=(A+a')+(B+b')+\dots-(n-2)\pi=n\pi-(n-2)\pi=2\pi=\frac{S}{r^2}+P.$$

[With a suitable definition of polar polygon, the same enunciation does for all cases.]

2706. (By J. GRIFFITHS, M.A.)—Prove that (1) two real equilateral hyperbolas can be drawn to touch the sides of a given obtuse-angled triangle, and to pass through the centre of its circumscribed circle; and (2) the centres of these curves are the points of intersection of the nine-point and circumscribed circles of the triangle.

Solution by Prof. WOLSTENHOLME, M.A.

When a rectangular hyperbola touches the sides of a given triangle, it is a well-known property that its centre lies on the polar circle of the triangle; also, when a conic is inscribed in a given triangle and passes through the centre of the circumscribed circle, it has been proved (in the Solution of Quest. 2726, on p. 60 of Vol. XI. of *Reprint*) that the director circle touches the circumscribed circle of the triangle; or, when the conic is a rectangular hyperbola, its centre lies on the circumscribed circle. Hence for a rectangular hyperbola touching the sides of a given triangle, and passing through the centre of the circumscribed circle, the centre will be the two common points of the circumscribed circle and polar circle, *i.e.*, of the circumscribed and nine-point circle, and these are real points (and therefore the hyperbolas real) when the triangle is obtuse-angled.

[It is interesting to notice how it happens there are only two solutions instead of four. The locus of the centre of a conic inscribed in a given triangle and passing through a given point, is a conic touching the straight lines joining the middle points of the sides of the given triangle; and this conic generally intersects the polar circle in four points; but when the given point is the centre of the circumscribed circle, these four coalesce

two and two, and the locus has double contact with the polar circle at points lying on the circumscribed and nine-point circles. The locus of the centre of a rectangular hyperbola touching the sides of the triangle of reference, is the polar circle $x^2 \cot A + y^2 \cot B + z^2 \cot C = 0$; and the locus of the centre of a conic inscribed in the triangle of reference and passing through the centre of the circumscribed circle, is the conic

$$\{(y+z-x)\sin 2A\}^{\frac{1}{2}} + \ldots + \ldots = 0, \text{ or}$$

$$x^2 \cot A + y^2 \cot B + z^2 \cot C - \sin A \sin B \sin C\,(x \cot A + y \cot B + z \cot C)^2 = 0,$$

a conic touching the polar circle at the two points where the straight line $x \cot A + y \cot B + z \cot C = 0$ meets it. But

$$x \cot A + y \cot B + z \cot C = 0$$

is the common radical axis of the circumscribed, nine-point, and polar circles of the triangle; the circumscribed and nine-point circles being

$$x^2 \cot A + y^2 \cot B + z^2 \cot C = (x+y+z)\,(x \cot A + y \cot B + z \cot C),$$

$$x^2 \cot A + y^2 \cot B + z^2 \cot C = \tfrac{1}{2}\,(x+y+z)\,(x \cot A + y \cot B + z \cot C).$$

The common points of these circles are real when $\cot A \cot B \cot C$ is negative; that is, when the triangle is obtuse-angled, or when the polar circle is real.]

6338. (By W. H. H. Hudson, M.A.) — From a semicircle, whose diameter is in the surface of a fluid, a circle is cut out, whose diameter is the vertical radius of the semicircle; find the centre of pressure of the remainder.

I. *Solution by* W. J. C. Sharp, M.A.; A. Anderson, B.A.; *and others*.

The centre of pressure is evidently in the vertical diameter, and the pressure on a strip of width dx at a depth x

$$\propto \left\{(r^2-x^2)^{\frac{1}{2}} - (rx-x^2)^{\frac{1}{2}}\right\} x\,dx,$$

where r is the radius of the semicircle; hence we have

$$\bar{x} = \frac{\int_0^r \left\{(r^2-x^2)^{\frac{1}{2}} - (rx-x^2)^{\frac{1}{2}}\right\} x^2\,dx}{\int_0^r \left\{(r^2-x^2)^{\frac{1}{2}} - (rx-x^2)^{\frac{1}{2}}\right\} x\,dx} = \frac{\dfrac{\pi r^4}{16} - \dfrac{5\pi r^4}{128}}{\dfrac{r^3}{3} - \dfrac{\pi r^3}{16}} = \frac{9\pi r}{8\,(16-3\pi)}.$$

II. *Solution by the* Proposer.

The depth of the centre of pressure of a plane area immersed vertically in a fluid is $k^2 \div h$, where k is the radius of gyration of the area about the line of floatation, h the depth of the centre of gravity. For the semicircle (radius r) $k^2 = \frac{1}{4}r^2$, $h = \dfrac{4r}{3\pi}$; and for the circle (radius $\frac{1}{2}r$) $k^2 = \frac{5}{16}r^2$, $h = \frac{1}{2}r$,

also, pressure on semi-circle : pressure on circle

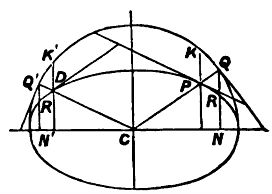
$$= \frac{\pi r^2}{2} \cdot \frac{4r}{3\pi} : \frac{\pi r^2}{4} \cdot \frac{r}{2} = 16 : 3\pi;$$

therefore depth of centre of pressure of remainder

$$= \frac{16 \,.\, \frac{3}{32}\pi - 3\pi \,.\, \frac{5}{8}\pi}{16 \,.\, - 3\pi} r = \frac{9\pi r}{8(16-3\pi)}.$$

6423. (By Rev. T. W. Openshaw, M.A.) — If a pair of conjugate diameters CP, CD of an ellipse be produced to meet the auxiliary circle in Q, Q′, and the ordinates QN, Q′N′ meet the ellipse in R, R′; prove (1) that tangents at R, R′ intersect on the director-circle: and (2) extend this also to the ellipsoid; also, if the ordinates of P and D meet the auxiliary circle at K, K′, prove (3) that the tangents to the auxiliary circle at K, K′ meet on the auxiliary circle of the ellipse on which the tangents at P and D intersect.

Solution by G. F. Walker, M.A.; N. Sarkar, B.A.; *and others.*



The relation $\tan\phi \tan\phi' = -\frac{b^2}{a^2}$ can be easily put in the form

$$a^2 \sin\phi \sin\phi' + b^2 \cos\phi \cos\phi' = 0,$$

$$a^2 [\cos^2 \tfrac{1}{2}(\phi-\phi') - \cos^2 \tfrac{1}{2}(\phi+\phi')]$$
$$+ b^2 [\cos^2 \tfrac{1}{2}(\phi-\phi') - \sin^2 \tfrac{1}{2}(\phi+\phi')] = 0;$$

giving $a^2\left(1-\frac{x^2}{a^2}\right) + b^2\left(1-\frac{y^2}{b^2}\right) = 0,$

or the auxiliary circle as the locus of the intersection of tangents at R and R′. The point of intersection of the tangents at K and K′, and the point of intersection of the tangents at P and D will lie on a line perpendicular to the major axis, and the latter will divide the perpendicular from the former in the ratio $b : a$ (segments measured from the major axis). The locus of the latter point is known to be an ellipse similar and similarly situated to the given ellipse, and therefore the tangents at K and K′ will meet in its auxiliary circle.

In the case of the ellipsoid, if l_1, m_1, n_1, &c. be the conjugate diameters, $\frac{al_1}{r_1}$, $\frac{bm_1}{r_1}$, &c. will be the other three points.

If (x, y, z) be the point of concourse of the tangents, we have

$$\frac{x}{a}\left| \frac{al_1}{r_1}, \frac{bm_1}{r_1}, \frac{cn_1}{r_1} \right| = \left| 1, \frac{bm_1}{r_1}, \frac{cn_1}{r_1} \right|,$$

and reducing by the known relations between the l, m, n's, we have

$$\frac{abcx}{ar_1r_2r_3} = bc\left(\frac{l_1}{r_2r_3}\dots\right), \text{ and } x = l_1r_1 + l_2r_2 + l_3r_3;$$

hence $x^2+y^2+z^2 = (l_1r_1 + l_2r_2 + l_3r_3)^2 \dots = r_1^2 + r_2^2 + r_3^2 = a^2+b^2+c^2$, and the locus is the director sphere.

6486. (By J. R. Wilson, M.A.)—From any point P on a central conic lines are drawn to the extremities of the axis BB′. BQ perpendicular to BP meets PB′ in Q, and B′R perpendicular to B′P meets PB in R. Show that the envelope of QR is a central conic, the ratio of whose axes is $a^2+b^2 : 2ab$.

Solution by R. Knowles, B.A., L.C.P.; W. J. C. Sharp, M.A.; *and others.*

Let the coordinates of P be $a\cos\phi$, $b\sin\phi$; then, forming the equations to PB, B′R; B′P, BQ, we find the coordinates of R and Q, and thence the equation to QR, which is

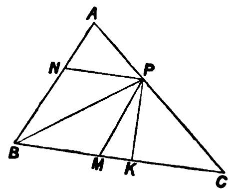
$$(a^4-b^4)\,y\cos\phi + 2ab\,(a^2-b^2)\,x\sin\phi + 2ab^2\,(a^2+b^2) = 0,$$

whose envelop is $4a^2b^2\,(a^2-b^2)^2\,x^2 + (a^4-b^4)^2\,y^2 = 4a^2b^4\,(a^2+b^2)^2$,

the semi-axes $b\,.\,\dfrac{a^2+b^2}{a^2-b^2}$ and $\dfrac{2ab^2}{a^2-b^2}$, and the areas as $a^2+b^2 : 2ab$.

6631. (By Prof. Matz, M.A.)—Show that the average area of parallelograms inscribed in a triangle is one-third of the area of the triangle.

Solution by C. Morgan, B.A.; Belle Easton; *and others.*



Let $CP = s$; then area of parallelogram NPMB

$$= PK\,.\,BM = s\,.\,\sin C\,.\,\frac{a}{b}\,(b-s);$$

hence the mean area is therefore

$$\frac{a}{b}\sin C\int_0^b s\,(b-s)\,ds \div b$$

$$= \frac{a}{b^2}\sin C\int_0^b \left[\tfrac{1}{2}b^3-\tfrac{1}{3}b^3\right]$$

$$= \tfrac{1}{6}ab\,.\sin C = \tfrac{1}{3} \text{ of the area of the triangle ABC.}$$

[See the Editor's Question 1217.]

6240. (By C. Leudesdorf, M.A.)—A homogeneous cube, whose edge is $2a$, strikes with one of its angular points against a perfectly rough inelastic wall. Just before impact the cube was moving in a given direction with velocity v, and rotating about an axis through its centre parallel to this direction with angular velocity ω. Prove that, if the centre of the cube begins, after the impact, to move with velocity v' in a direction making an angle θ with its direction before impact, then

$$\cos\theta = \frac{33vv'}{4a^2\omega^2+27v^2}.$$

Solution by G. F. WALKER, M.A.; D. EDWARDES; *and others.*

Let the point of impact be taken as origin, and the three edges which meet in that point as axes of coordinates. Let I be the moment of inertia of the cube about a line through its centre of gravity; α, β, γ the angles which the direction before impact makes with the axes, ω_x, ω_y, ω_z the angular velocities after impact about the axes. The velocity of the centre after impact along the axis of x will be therefore $a(\omega_y-\omega_z)$, and similarly for the axes of y and z. Hence

$$\cos\theta = av^{-1}\left\{\cos\alpha\,(\omega_y-\omega_z)+\cos\beta\,(\omega_z-\omega_x)+\cos\gamma\,(\omega_x-\omega_y)\right\},$$

and the dynamical equations will be, with two similar equations,

$$\mathrm{I}\omega\cos\alpha + av(\cos\gamma-\cos\beta) = (\mathrm{I}+2a^2)\,\omega_x - a^2(\omega_y+\omega_z);$$

hence $$\cos\theta = \frac{a^2v\,(2-2\cos\alpha\cos\beta-2\cos\beta\cos\gamma-2\cos\gamma\cos\alpha)}{(\mathrm{I}+3a^2)\,v'}.$$

Also, subtracting the equations, squaring, and adding, we get

$$(\mathrm{I}+3a^2)^2\frac{v'^2}{a^2} = (2-2\cos\alpha\cos\beta-2\cos\beta\cos\gamma-2\cos\gamma\cos\alpha)\,(\mathrm{I}^2\omega^2+3a^2v^2);$$

therefore $\cos\theta = \dfrac{(\mathrm{I}+3a^2)\,vv'}{\mathrm{I}^2\omega^2+3a^2v^2}$; and, putting $\mathrm{I}=\frac{2}{3}a^2$, we have &c.

6029. (By W. J. C. SHARP, M.A.)—Prove that the covariant

$$\mathrm{A}\alpha^2+\mathrm{B}\beta^2+\mathrm{C}\gamma^2+2\mathrm{F}\beta\gamma+2\mathrm{G}\gamma\alpha+2\mathrm{H}\alpha\beta$$

is the locus of points whose polar conics are parabolas, and separates the points whose polar conics are ellipses from those whose polar conics are hyperbolas.

Solution by the PROPOSER.

If $\alpha x+\beta y+\gamma z=0$ be the equation to the line at infinity, the conic $ax^2+by^2+cz^2+2fyz+2gxz+2hzy=0$ will meet it in two real, identical, or imaginary points, according as $\mathrm{A}\alpha^2+\mathrm{B}\beta^2+$ &c. is negative, zero, or positive; that is, the polar conic is an hyperbola, a parabola, or an ellipse according to the sign of $\mathrm{A}\alpha^2$, &c., which proves the proposition.

6453. (By J. L. MCKENZIE, B.A.)—Three consecutive numbers may readily be found, each of which contains a square factor > 1; *e.g.*, 48, 49, 50; 98, 99, 100; 124, 125, 126; &c. Is the same possible for four consecutive numbers? If so, find the first case in which it occurs.

Solution by G. HEPPEL, M.A.; C. BICKERDIKE; *and others.*

It is possible, and the next set of three occurring after those mentioned forms part of a sequence of four, the numbers being 242, 243, 244, 245.

6272. (By E. Anthony, M.A.)—The normal at any point P of a curve intersects two straight lines, which meet in the point O, and which are at right angles to one another, in the points G and g, so that we have

$$m \,.\, OG^2 \,.\, Pg = n \,.\, Og^2 \,.\, PG\text{; find the equation to the curve.}$$

Solution by C. Morgan, B.A ; J. Young, B.A. ; *and others.*

Taking the straight lines as axes of coordinates, the intercepts on them by the normal are $\dfrac{x\,dx+y\,dy}{dx}$, $\dfrac{x\,dx+y\,dy}{dy}$; therefore $\dfrac{Og^2}{OG^2}=\left(\dfrac{dx}{dy}\right)^2$. And $PG=y\dfrac{ds}{dx}$, $Pg=x\dfrac{ds}{dy}$; therefore $\dfrac{Pg}{PG}=\dfrac{x}{y}\dfrac{dx}{dy}$. Hence, by the given relation, $\dfrac{dx}{dy}=0$, whence $x=a$ or $\dfrac{dx}{dy}=\dfrac{m}{n}\dfrac{x}{y}$, therefore $y^m=Cx^n$.

6345. (By Prof. Casey, F.R.S.)—A uniform circular plate is placed with its centre upon a prop; find at what points on its circumference three given weights must be attached in order that it may rest in a horizontal position.

Solution by D. Edwardes ; J. Hayash, M.A. ; *and others.*

If α, β, γ be the angles subtended at the centre by the weights in pairs; then, taking moments about a vertical plane through the centre and the weight w_3, we have $w_1 \sin\beta = w_2 \sin\alpha$; hence

$$w_1 : w_2 : w_3 = \sin\alpha : \sin\beta : \sin\gamma,$$

and α, β, γ are consequently the complements of the angles of a triangle whose sides are proportional to the weights.

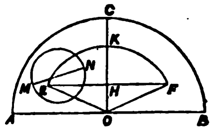
5592. (By J. Dawson.)—Given one point in the circumference of a circle, and two points whose respective distances from the centre are the m^{th} and n^{th} parts of the radius; show how to draw the circle.

Solution by Prof. Morel ; Prof. Matz, M.A. ; *and others.*

Le centre de ce cercle se trouve sur deux lieux géométriques faciles à déterminer. Soit A le point sur la circonférence, B et C les deux autres points; on a, O étant le centre, $OB : OA = 1 : m$; donc le point O est sur une circonférence de cercle que l'on trace immédiatement. De même $OC : AO = 1 : n$; donc O se trouve sur une seconde circonférence de cercle. Il en résulte qu'il est à leur point de rencontre. On a ensuite le rayon en joignant le point O au point A.

5651. (By Prof. SEITZ, M.A.)—If a circle be drawn on the line joining two points taken at random in the surface of a given semicircle; show that the chance that the circle lies wholly within the semicircle is $\frac{4}{3} - \frac{128}{45\pi}$.

Solution by the PROPOSER.



Let ABC be the given semicircle; O its centre; M, N the random points. With O as a centre, and a radius equal to the difference between the given radius and the half of MN, draw the arc EKF, the chord EF being at a distance from AB equal to the half MN; and draw OH perpendicular to EF. Now, while MN is given in length and direction, the area of the segment EKF represents the number of ways the two points can be taken, so that the circle on MN will be wholly within the semicircle.

Let MN $= 2x$, AO $= r$, $\theta =$ the angle which MN produced makes with AB. Then OH $= x$, OE $= r-x$, and

$$\text{area of segment EKF} = (r-x)^2 \cos^{-1}\left(\frac{x}{r-x}\right) - x\,(r^2-2rx)^{\frac{1}{2}}.$$

An element of the semicircle at N is $4x\,dx\,d\theta$; also the limits of x are 0 and $\frac{1}{2}r$; and of θ, 0 and 2π. The whole number of ways the two points can be taken is $\frac{1}{4}\pi^2 r^4$. Hence we have

$$p = \frac{4}{\pi^2 r^4}\int_0^{\frac{1}{2}r}\int_0^{2\pi}\left[(r-x)^2\cos^{-1}\left(\frac{x}{r-x}\right) - x\,(r^2-2rx)^{\frac{1}{2}}\right]4x\,dx\,d\theta$$

$$= \frac{32}{\pi r^4}\int_0^{\frac{1}{2}r}\left[(r-x)^2\cos^{-1}\left(\frac{x}{r-x}\right) - x\,(r^2-2rx)^{\frac{1}{2}}\right]x\,dx = \frac{4}{3} - \frac{128}{45\pi}.$$

6615. (By H. McCOLL, B.A.)—Speaking of the limits of multiple integrals, if z_r (the r^{th} limit of z) be less than z_m and greater than z_n, we have the statemental equation $z_{m'n} = z_{m'r} + z_{r'n}$. In what sense is this equation true when z_r is not thus restricted?

Solution by W. B. GROVE, B.A.; E. H. HAIGH, B.A., B.Sc.; *and others.*

In $z_{m'.n} = z_{m'.r} + z_{r'.n}$ the letters denote statements as to the upper and lower limits of integration. Let $(z_{m'.n})$ denote the value of the integral taken between those limits, then the equation $(z_{m'.n}) = (z_{m'.r}) + (z_{r'.n})$ is universally true. For suppose $r < n$, then

$$(z_{r'.n}) = -(z_{n'.r}) \text{ and } (z_{m'.r}) = (z_{m'.n}) + (z_{n'.r}),$$

therefore $$(z_{m'.r}) + (z_{r'.n}) = (z_{m'.n}).$$

6613. (By G. S. Carr, B.A.)—If A, B, C are the normal, and F, G, H the tangential stresses upon a rectangular element of a strained elastic solid, show—(1) that the resultant tangential stress, at the same point, upon a plane whose direction-cosines with respect to the axes of A, B, C are l, m, n, is $\left[\left\{F(m^2-n^2)+(Gm-Hn)\,l-(B-C)\,mn\right\}^2+\text{\&c.}\right]^{\frac{1}{2}}$;
and (2) that its direction-cosines are proportional to

$$l\left\{A(m^2+n^2)-Bm^2-Cn^2-2mnF\right\}+(m^2+n^2-l^2)(Gn-Hm),\ \text{\&c.}$$

Solution by D. Edwardes; Lizzie A. Kittudge; *and others.*

(1) Let F′ denote the stress in question, and let P, Q, R be the components, in the direction of the axes, of the stress on the plane; then the stress normal to the plane is $lP+mQ+nR$. Hence

$$P^2+Q^2+R^2=(lP+mQ+nR)^2+F'^2,$$

or $$F'^2=(mR-nQ)^2+(nP-lR)^2+(lQ-mP)^2.$$

Now $P=Al+Hm+Gn,\ \ Q=lH+mB+nF,\ \ R=lG+mF+nC,$

in the small strain; wherefore, substituting in the above expression,

$$F'^2=[F(m^2-n^2)+(Gm-Hn)\,l-(B-C)\,mn]^2+\text{\&c.}+\text{\&c.}$$

(2) Let X be the x-component of the stress F′. Then, projecting along the axis of x, $P=l(lP+mQ+nR)+X$, or $X=P(m^2+n^2)-lmQ-nlR$; substituting for P, Q, R their values given above, we have

$$X=l[A(m^2+n^2)-Bm^2-Cn^2-2mnF]+(m^2+n^2-l^2)(mH+nG),$$

and similarly for the other components.

6577. (By the Rev. T. R. Terry, M.A.)—If a triangle PQR be inscribed in a parabola, so that PQ, PR are the normals at Q, R; show that (1) the centre of gravity of the triangle lies on the axis of the parabola; and (2) the side QR passes through a fixed point on the axis produced at a distance from the vertex equal to the semi-latus rectum.

Solution by E. W. Symons, M.A.; R. Knowles, B.A., L.C.P.; *and others.*

Q and R, the feet of normals from P (x', y'), are two of the intersections of the curves $y^2-4ax=0$, $xy+y(2a-x')-2ay'=0$, and also $y'^2=4ax'$. Eliminating x, we get a cubic for y with no square term, therefore centre of gravity of the three points P, Q, R lies on $y=0$.

Again, if $lx+my+n=0$ be equation of QR,

$$K(y^2-4ax)+(lx+my+n)(y-y')=0$$

can be made identical with $xy+y(2a-x')-2ay'=0$. This gives $K+m=0$, $l=\dfrac{n}{2a}$, &c. &c.; therefore QR is $n(x+2a)+2amy=0$, which passes through fixed point $(-2a, 0)$; therefore, &c.

6655. (By G. F. Walker, M.A.)—**Show that, according as q is an integer numerically greater than or less than p,**

$$\int_0^{\frac{1}{2}\pi} \cos^{2p}\theta \cos^2 q\theta \, d\theta = \frac{\pi}{2^{2p+2}} \frac{(2p)!}{(p!)^2},$$

or
$$\frac{\pi}{2^{2p+2}} \left\{ \frac{(2p)!}{(p!)^2} + \frac{(2p)!}{(p+q)!\,(p-q)!} \right\}.$$

Solution by the Rev. T. R. Terry, M.A.; Prof. Matz, M.A.; *and others.*

$$\mathrm{I} = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \cos^{2p}\theta\,(1+\cos 2q\theta)\,d\theta = \tfrac{1}{2} \cdot \frac{(2p)!}{2^{2p}\,.\,p\,!\,p\,!}\,\tfrac{1}{2}\pi + \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \cos^{2p}\theta \cos 2q\theta\, d\theta.$$

Now $\qquad 2^{2p}\cos^{2p}\theta = 2\,[\cos 2p\theta + 2p\cos 2\,(p-1)\,\theta + \dots].$

If $q > p$, obviously $\cos 2q\theta$ cannot occur in this series,

therefore $\qquad \displaystyle\int_0^{\frac{1}{2}\pi} \cos^{2p}\theta \cos 2q\theta\, d\theta = 0.$

If $p > q$, then $\cos 2q\theta$ does occur, and its coefficient $= \dfrac{(2p)!}{(p-q)!\,(p+q)!}$.

Therefore $\displaystyle\int_0^{\frac{1}{2}\pi} \cos^{2p}\theta \cos 2q\theta\, d\theta = \frac{(2p)!}{(p-q)!\,(p+q)!} \cdot \frac{1}{2^{2p}} \cdot \tfrac{1}{2}\pi$; therefore, &c.

6262. (By the Rev. H. G. Day, M.A.)—**If a line of length c is divided into n parts, $x, y, z, \dots$, by $(n-1)$ points taken at random in it, prove that the average values (1) of xyz, (2) of $x^{\alpha}y^{\beta}z^{\gamma}$ are**

$$\frac{(n-1)!}{(2n-1)!}c^n \text{ and } \frac{(n-1)!\,\alpha!\,\beta!\,\gamma!\dots}{(n-1+\alpha+\beta+\gamma+\dots)!}c^{\alpha+\beta+\gamma+\dots}$$

Solution by the Proposer.

Let $u_n c^n$ be the required quantity; also let x be the first part; hence, since the remainder is divided into $(n-1)$ parts at random, $u_n c^n$ = average value of $x\,(c-x)^{n-1}\,u_{n-1}$, x being taken according to conditions prescribed;

therefore $\qquad \displaystyle u_n c^n = \int_0^c x u_{n-1}\,(c-x)^{n-1}\left(\frac{c-x}{c}\right)^{n-2}(n-1)\,\frac{dx}{c},$

$$u_n = \int_0^1 (n-1)\,x\,(1-x)^{2n-3}\,dx\,u^{n-1} = \frac{(n-1)\,u_{n-1}}{(2n-1)\,(2n-2)} = \frac{(n-1)!}{(2n-1)!}.$$

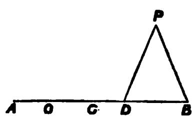
2. If $x, y, z \dots$ are the successive parts, the average value of

$$x^{\alpha}y^{\beta}z^{\gamma}\dots \text{ is } \frac{(n-1)!\,\alpha!\,\beta!\,\gamma!\dots}{(n-1+\alpha+\beta+\gamma+\dots)!}c^{\alpha+\beta+\gamma+\dots}.$$

[Mr. Day remarks that, though he has more than once been on the point of admitting a fallacious Solution, he can guarantee the correctness of the values of the chances given above.]

5405. (By S. TEBAY, B.A.) — A straight rod, resting on a smooth horizontal plane, is struck by a random shot from a given point in the plane; find the mean angular velocity of the rod, and the mean motion of the centre of gravity of the rod.

Solution by the PROPOSER.



Let AB be the rod, bisected in C, P the point from which the shot is fired, D the point of impact, and O the instantaneous centre of rotation. Let $AB = 2a$, $BP = b$, $CD = h$, $OC = x$, $\angle ABP = \alpha$, $BPD = \theta$, ω the angular velocity of the rod, m its mass, m' the mass of the shot, u its velocity, and F the effect of the blow at D. Then we have

$$(x^2 + \tfrac{1}{3}a^2)\, m\omega = F\,(x+h), \quad mx\omega = F, \quad \text{from which } \tfrac{1}{3}a^2 m\omega = Fh.$$

But $$h = a - \frac{b\sin\theta}{\sin(\alpha+\theta)}, \quad F = m'u\sin(\alpha+\theta);$$

therefore $$\tfrac{1}{3}a^2 m\omega = m'u\left\{a\sin(\alpha+\theta) - b\sin\theta\right\}.$$

If the shot be fired between the limits $\theta = \gamma$, $\theta = \beta$, the mean angular

velocity $$= \frac{1}{\beta-\gamma}\int_\gamma^\beta \omega\, d\theta$$

$$= \frac{3m'u}{a^2 m(\beta-\gamma)} \sin\tfrac{1}{2}(\beta-\gamma)\left\{a\sin\tfrac{1}{2}(2\alpha+\beta+\gamma) - b\sin\tfrac{1}{2}(2\alpha+\beta+\gamma)\right\}.$$

Let V be the velocity of the centre of gravity; then

$$V = \frac{F}{m} = \frac{m'u}{m}\sin(\alpha+\theta);$$

therefore $$\frac{1}{\beta-\gamma}\int_\gamma^\beta V d\theta = \frac{m'u}{m(\beta-\gamma)}\left\{\cos(\alpha+\gamma) - \cos(\alpha+\beta)\right\}.$$

6696. (By D. EDWARDES.)—If $m\sin(\theta+\phi) = \cos(\theta-\phi)$, prove that $(1-m\sin 2\theta)^{-1} + (1-m\sin 2\phi)^{-1} = 2(1-m^2)^{-1}$.

Solution by Prof. SCOTT, M.A.; J. O'REGAN; *and others.*

If $m\sin(\theta+\phi) = \cos(\theta-\phi)$, then $\tan\phi = \dfrac{1-m\tan\theta}{m-\tan\theta}$; thence

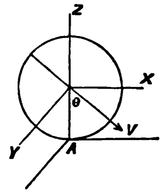
$$\sin 2\phi = \frac{2m-(1+m^2)\sin 2\theta}{1+m^2-2m\sin 2\theta}, \text{ and } (1-m\sin 2\phi)^{-1} = \frac{1+m^2-2m\sin 2\theta}{(1-m^2)(1-m\sin 2\theta)},$$

which gives $$(1-m\sin 2\theta)^{-1} + (1-m\sin 2\phi)^{-1} = \frac{2}{1-m^2}.$$

6459. (By C. LEUDESDORF, M.A.)—A homogenous sphere of mass m and radius a is rotating about a diameter with angular velocity Ω while its centre is moving in the direction of this diameter with velocity V, when it strikes a perfectly rough horizontal plane. Show that the kinetic energy lost by the impact is $\frac{1}{2}mV^2(1-e^2)\cos^2\theta+\frac{1}{7}m(V^2+a^2\Omega^2)\sin^2\theta$, θ being the angle of incidence, and e the coefficient of restitution.

Solution by G. S. CARR, B.A.; D. EDWARDES; *and others.*

Take rectangular axes as in the figure, the plane XZ to contain the axis of the rotating sphere, and also its point of contact A with the rough plane.



The first term is the *vis viva* lost through imperfect elasticity, and is a familiar result. The second expresses the amount lost through the tangential reaction at A. To find this, let u be the velocity of the point A resolved parallel to the plane, and which is wholly destroyed by the impact. By resolving V and Ω along the X and Y axes, we get

$$u^2 = V^2\sin^2\theta + a^2\Omega^2\sin^2\theta.$$

Now, if an impulse P applied tangentially to a sphere generates a velocity u at the surface, v at the centre, and an angular velocity ω, we know from the equations $u = v + a\omega$, $P = mv$, and $mk^2\omega = Pa$, that $v = \frac{2}{7}u$ and $a\omega = \frac{5}{7}u$. Hence the *vis viva* generated is $\frac{1}{2}m(v^2+k^2\omega^2) = \frac{1}{7}mu^2$. This is therefore the *vis viva* lost, in the present case, by destroying the velocity u.

6583. (By W. J. C. SHARP, M.A.)—Show that the equation $\left(\frac{1+p^2}{q}\frac{d}{dx}\right)^n \cdot \frac{(1+p^2)^{\frac{3}{2}}}{q} = 0$ represents any $(n-1)^{\text{th}}$ involute of a circle.

Solution by E. W. SYMONS, M.A.; J. HAMMOND, M.A.; *and others.*

If $\rho = f(\phi)$ be equation to a curve, ρ being radius of curvature, and ϕ the angle which tangent makes with a fixed line, we know that $\frac{d\rho}{d\phi}$ is radius of curvature of corresponding point on evolute, and generally $\frac{d^n\rho}{d\phi^n}$ is radius of curvature of point on n^{th} evolute. Now, if $\rho = f(\phi)$ be the $(n-1)^{\text{th}}$ involute of a circle, its n^{th} evolute is a point, viz. the centre of the circle, and radius of curvature of a point is $= 0$; therefore equation of $(n-1)^{\text{th}}$ involute is

$$\frac{d^n\rho}{d\phi^n} = 0, \text{ or } \left(\frac{d}{d\phi}\right)^n \cdot \rho = 0,$$

or $$\left(\frac{dx}{d\phi}\cdot\frac{d}{dx}\right)^n \cdot \rho = 0, \text{ or } \left(\frac{1+p^2}{q}\cdot\frac{d}{dx}\right)^n \cdot \frac{(1+p^2)^{\frac{3}{2}}}{q} = 0.$$

6575. (By H. McColl, B.A.)—Let S denote any multiple integral; αS the value of S when the integration is restricted by the statement α; $(\alpha-\beta+\gamma-\ldots)$ S an abbreviation for $\alpha S-\beta S+\gamma S-\ldots$; and $\alpha(1-\beta)(1-\gamma)S$ an abbreviation for $(\alpha-\alpha\beta-\alpha\gamma+\alpha\beta\gamma)$ S, and so on; then show that $\alpha\beta'\gamma'\ldots\ S=\alpha(1-\beta)\ (1-\gamma)\ \ldots\ S$. [Mr. McColl states that this simple theorem will often effect great simplification in dealing with the limits of multiple integrals].

Solution by W. B. Grove, B.A.

$$\text{Since } \alpha S=\alpha(\beta+\beta')(\gamma+\gamma')S=(\alpha\beta\gamma+\alpha\beta\gamma'+\alpha\beta'\gamma+\alpha\beta'\gamma')S,$$

$$\therefore\ \alpha\beta'\gamma'S=(\alpha-\alpha\beta-\alpha\beta'\gamma)S=(\alpha-\alpha\beta-\alpha\gamma+\alpha\beta\gamma)S=\alpha(1-\beta)(1-\gamma)S.$$

In fact, these integrals can be treated in exactly the same way as the combinations in Prof. Jevons' *Numerical Logic*, in which

$$(ab'c')=(a)-(ab)-(ac)+(abc).$$

6528. (By R. B. Hayward, M.A.)—In a "three-cornered" constituency (*i.e.*, one which returns three members) each voter has two votes, but cannot give both to the same candidate. Supposing the majority, consisting of M voters, to put forward three candidates, and the minority, consisting of m voters, to put forward two, and supposing that all the voters take part in the election and give both their votes; prove that (1) the chance that the three candidates of the majority will be all elected is $1-3\dfrac{(m+1)(m+2)}{(M+1)(M+2)}$ or $\dfrac{(2M-3m-1)(2M-3m-2)}{(M+1)(M+2)}$, according as m is $<$ or $>$ $\frac{1}{2}$M, and $=\frac{1}{4}$ nearly if $m=\frac{1}{2}$M; and (2) that, if $m=\frac{2}{3}$M, the chance that two of the three will be defeated is $\frac{1}{8}$ nearly.

Solution by G. F. Walker, M.A.; Prof. Matz, M.A.; *and others*.

Let A, B, C be the majority candidates, and let x denote the number who vote for B and C, y ... C and A, z ... A and B. Then $x+y+z=$ M. Also, if A, B, and C are to be elected, $y+z=>m$, $z+x=>m$, $x+y=>m$

Take an equilateral triangle of reference, and trace the lines

$$(y+z)M=m(x+y+z),\ \&c.$$

If $M>2m$, we have Fig. 1, and if $M<2m$, we have Fig. 2. The required chance is the number of points which fall in area (p) in either figure.

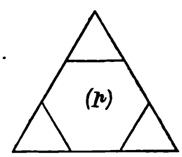


Fig 1.

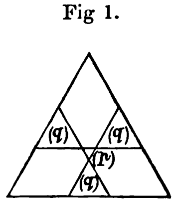


Fig. 2.

In the first case, it is best to find number in either of the smaller triangles. Thus, as

$$1+2+\ \ldots\ m+1=\tfrac{1}{2}(m+1)(m+2),$$

the chance is

$$1-\left\{3\left[\tfrac{1}{2}(m+1)(m+2)\right]\div\tfrac{1}{2}(M+1)(M+2)\right\}$$
$$=1-3\frac{(m+1)(m+2)}{(M+1)(M+2)}.$$

In the second case, the number is equal to

$$1+2+\dots[(m+1)-2(m-M)] = \tfrac{1}{2}(2M-3m+1)(2M-3m+2),$$

and the required chance is $\dfrac{(2M-3m+1)(2M-3m+2)}{(M+1)(M+2)}$.

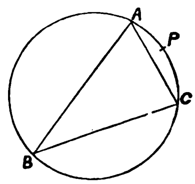
The chance that two of the three are rejected, is the sum of the number of points in the areas (q), to whole numbers, and the chance is

$$3\frac{(2m-M+1)(2m-M+2)}{(M+1)(M+2)}.$$

The special chances are easily found by substituting for m, and supposing M large, as would usually be the case in such a constituency.

6688. (By C. LEUDESDORF, M.A.)—If a, b, c be three quantities such that any two are together greater than the third, and if x, y, z be three quantities whose sum is positive; show that, if $a^2x^{-1}+b^2y^{-1}+c^2z^{-1}=0$, the product xyz must be negative.

I. *Solution by* E. W. SYMONS, M.A.; J. O'REGAN; *and others.*



a, b, c are the sides of a triangle; x, y, z are proportional to the triangular coordinates of a point with respect to it; and because $a^2x^{-1}+\dots=0$, this point lies on the circumscribed circle of the triangle; therefore the point (x, y, z) lies outside the triangle, and within one of its angles; and therefore one of the three x, y, z is negative; therefore &c.

II. *Solution by* R. TUCKER, M.A.; A. MARTIN, M.A.; *and others.*

$$x+y+z=k \text{ (positive)}:$$

therefore $\quad 0 = a^2y(k-x-y)+b^2x(k-x-y)+c^2xy,$

i.e., $\quad (a^2y+b^2x)k = (a^2+b^2-c^2)xy+a^2y^2+b^2x^2;$

therefore $\quad -c^2xyk = [(a+b)^2-c^2]xyz+(ay-bx)^2z$

$\qquad$ or $\quad = (ay+bx)^2z-[c^2-(a-b)^2]xyz \quad\dots\dots\dots\dots(1, 2).$

From (1), if xy be positive, z cannot be positive; from (2), if xy be negative, z cannot be negative; therefore xyz is negative.

6406. (By Prof. TOWNSEND, F.R.S.)—The equation of a conic in terms of the three perpendiculars λ, μ, ν on a variable tangent from the three vertices A, B, C of a fixed triangle self-reciprocal with respect to the curve,

being given in the form $l\lambda^2+m\mu^2+n\nu^2=0$, where l, m, n are constants; find, in terms of the same coordinates, that of the point-pair in which the axes of the curve intersect the line at infinity.

Solution by the PROPOSER.

Since evidently, for every diameter of the curve, $l\lambda+m\mu+n\nu=0$, which accordingly is the tangential equation of its centre, and since consequently, for any arbitrary line in its plane, the perpendicular distance from its centre $=(l\lambda+m\mu+n\nu)\div(l+m+n)$, we have therefore to find the relation of the second order connecting λ, μ, ν, for which the sum $(l\lambda+m\mu+n\nu)$ shall have its greatest and least values, subject to the given condition of tangency with the conic, viz. $l\lambda^2+m\mu^2+n\nu^2=0$, and to the general relation connecting the three perpendiculars from the three vertices of the triangle upon any arbitrary line in the plane, viz.

$$a^2\,.\,\lambda^2+b^2\,.\,\mu^2+c^2\,.\,\nu^2-2bc\cos\alpha\,.\,\mu\nu-2ca\cos\beta\,.\,\nu\lambda-2ab\cos\gamma\,.\,\lambda\mu=4\Delta^2,$$

where a, b, c are the three sides, α, β, γ the three angles, and Δ the area of the triangle; hence, differentiating with respect to the three variables, the sum and the two equations, and eliminating the three differentials, we get at once the determinantal equation

$$a\,(a\,.\,\lambda-c\cos\beta\,.\,\nu-b\cos\gamma\,.\,\mu)\,mn\,(\mu-\nu)+\&c.=0,$$

which accordingly is that of the point-pair in question.

Arranged in powers and products of the variables, this equation becomes

$$p\lambda^2+q\mu^2+r\nu^2-2\,(s-p)\,\mu\nu-2\,(s-q)\,\nu\lambda-2\,(s-r)\,\lambda\mu=0,$$

where $al\,(cm\cos\beta-bn\cos\gamma)=p$, &c., and $p+q+r=2s$; from which it appears that the point-pair in question is, as it ought to be, harmonically conjugate with respect to the point-pair

$$a^2\,.\,\lambda^2+b^2\,.\,\mu^2+c^2\,.\,\nu^2-2bc\cos\alpha\,.\,\mu\nu-2ca\cos\beta\,.\,\nu\lambda-2ab\cos\gamma\,.\,\lambda\mu=0,$$

which represents, to the same coordinates, the two circular points at infinity.

When $l:m:n=a\sec\alpha:b\sec\beta:c\sec\gamma$, that is, when the centre of the conic is the orthocentre of the triangle, and when the conic itself is consequently the orthocentric circle of the triangle, the preceding equation becomes, as it ought, indeterminate; every line through its centre being then an axis of the curve.

6502 & 6589. (By S. TEBAY, B.A.)—(6502). The letters in the annexed square represent any 16 numbers in arithmetical progression, the sum of the two extremes being s. They are so arranged that $2s$ is the sum of each of the 24 groups *abcd*, *efgh*, *ijkl*, *mnpq*, *aeim*, *bfjn*, *cgkp*, *ahlq*, *afkq*, *dgjm*, *abef*, *begf*, *cdgh*, *efij*, *fgjk*, *ghkl*, *klpq*, *ijmn*, *jknp*, *acik*, *bdjl*, *egmp*, *fhnq*, *admq*. Find in how many essentially different ways the numbers can be thus arranged, exclusive of all reversions, such as *dcba*, *miea*, &c.; and also the total number of ways in which four of these numbers make up $2s$.

a	*b*	*c*	*d*
e	*f*	*g*	*h*
i	*j*	*k*	*l*
m	*n*	*p*	*q*

(6589). The letters in the annexed square represent the first sixteen consecutive numbers, so arranged that each of the sixteen groups *abef*,

$cdgh$, $ijmn$, $klpq$, $fgjk$, $admq$, $abcd$, $efgh$, $ijkl$, $mnpq$, $aeim$, $bfjn$, $cgkp$, $dhlq$, $afkq$, $dgjm$ makes 34. There are 432 essentially different arrangements of these numbers, but a demonstration of the following property, with other allied properties, is still a desideratum. If $a+c = 17$, prove that $e+g = 17$.

Solution by G. Heppel, M.A.

Answering, first of all, the last part of Quest. 6502, there are 86 different sets of four numbers making up $2s$. The progression being $u+v$, $u+2v$, &c. ... $u+16v$, it is easy to see that, by taking u from each, and dividing by v, the case reduces to that of the first 16 natural numbers. Of these there are 19 of the required sets where 1 is the least number, 19 where 2 is the least, 18 where 3, 15 where 4, 10 where 5, 4 where 6, and 1 where 7. This gives a total of 86.

The case in Quest. 6502 is included in the more general Quest. 6589. Considering the latter, the following preliminary properties must be proved.

$$a+f+k+q = 34,\quad k+q+p+l = 34,\quad \therefore a+f = p+l, \text{ and } b+e+p+l = 34,$$

$$a+b+c+d = 34,\quad a+d+m+q = 34,\quad \therefore b+c = m+q,$$

and $\quad b+c+n+p = 34$. Hence also $e+l = c+n$,

$$e+f+g+h = 34,\quad f+g+j+k = 34,\quad \therefore e+h = j+k,$$

$$a+b+c+d = 34,\quad c+d+g+h = 34,\quad \therefore a+b = g+h,$$

$$a+m+d+q = 34,\quad m+j+g+d = 34,\quad \therefore a+q = j+g,$$

$$a+q = j+g, \text{ and } i+j = p+q,\quad \therefore a+i = g+p,$$

$$\therefore a+i+c+k = 34,\quad i+c = f+q,\quad a+k = h+n.$$

These equalities are merely types of several similar ones of each kind.

Call a set of four of the numbers necessarily making up 34 a *range*. Let two numbers of such a set be said to be in range with one another. Then, first it is necessary to prove that, if two numbers in range make up 17, there are eight pairs symmetrically situated each making up 17. The several cases must be considered separately.

First, let $a+b = 17$. This pairs off a, b, c, d, e, f, g, h.
If $j+i = 17$ there is symmetry.

$$\left.\begin{array}{lll} j+k = 17 & \text{gives} & e+h = 17, \\ j+l = 17 & ,, & b+d = 17, \\ j+m = 17 & ,, & c+h = 17, \\ j+n = 17 & ,, & b+f = 17, \end{array}\right\} \text{all inconsistent with the hypothesis.}$$

If $j+p = 17$, the only possibility for i is $i+q = 17$.
If $j+q = 17$, ,, ,, $i+p = 17$.
In either case $i+j+p+q = 34$, and $i+j = p+q$; therefore $i+j = 17$, which is inconsistent; therefore there must be symmetry.

Secondly, let $a+f = 17$. This pairs off a, f, b, e, k, l, p, q. If $d+g = 17$ there is symmetry. $d+c = 17$, $d+h = 17$, $d+m = 17$, $d+j = 17$, lead to impossibilities. $d+i = 17$ and $d+n = 17$ are similarly situated, so that one only need be considered. Suppose $d+i = 17$, then the only pair for n is $n+g = 17$, for $n+c = 17$ gives $b+p = 17$. Hence $d+i+n+g = 34$, and $d+g = i+n$; therefore $d+g = 17$, which is inconsistent.

Thirdly, let $a+k = 17$. This pairs off a, k, c, i, f, q, h, n. If $d+j = 17$, there is symmetry. If not, $d+e = 17$ and $d+p = 17$ are alone possible and are similar. Suppose $d+e = 17$, then must $j+p = 17$; therefore $d+e+j+p = 34$, $d+j = e+p$; therefore $d+j = 17$, which is inconsistent.

Fourthly, let $a+d=17$. This pairs off a, d, b, c, m, q, n, p. If $e+h=17$, there is symmetry. If not, we must have either $e+k=17$, $h+j=17$, or $e+j=17$, $h+k=17$; whence $j+k=17$, which is inconsistent.

Fifthly, let $a+c=17$. This pairs off a, b, c, d, i, j, k, l. If $m+p=17$, there is symmetry. If not, then either $m+f=17$, $p+h=17$, or $m+h=17$, $p+f=17$; whence $m+p=17$, which is inconsistent.

Sixthly and lastly, let $a+q=17$. This pairs off a, q, d, m, f, k, g, j, and we may have the symmetrical combination $a+q=17$, $m+d=17$, $f+k=17$, $g+j=17$, $b+p=17$, $c+n=17$, $e+l=17$, $i+h=17$. If not, then must either $e+c=17$, $n+l=17$, or $e+n=17$, $c+l=17$. This leads to four symmetrical arrangements, two of which it will be sufficient to consider, as the remaining two are similar to them.

$$\text{I. } e+c=17, \quad b+i=17, \quad n+l=17, \quad p+h=17.$$

$$\text{II. } e+c=17, \quad b+h=17, \quad n+l=17, \quad p+i=17.$$

In case I. $\quad a+b+c+e+f+g+i+j+k=68+a,$

$$d+h+l+q+m+n+p=68-q.$$

Therefore, adding, $136+a-q=136$, therefore $a=q$, which is impossible.

In case II.	$e+b+c+h=34$,	therefore $b+c=f+g$;
therefore	$b+c+j+k=34$,	therefore $a+i=b+j$.
Now	$e+c+i+p=34$,	therefore $e+i=g+k$;
therefore	$e+i+f+j=34$,	therefore $a+b=i+j$.

But it was shown that $a+i=b+j$, therefore $a=j$, which is impossible; therefore $a+q=17$ gives only the symmetrical arrangements first mentioned.

Every range is one of a set of four independent ranges together comprising the whole 16 numbers. Thus *bepl* is one of the set *bepl*, *chin*, *afkq*, *dgjm*.

Every range must contain two even and two odd numbers. For, if one range be all odd, another must be all even, and *vice versâ*. The remaining two independent ranges cannot be all odd and all even respectively, for then the sum of the two even ranges would be 72, and of the two odd ranges 64, which is impossible. And if the remaining two consist each of two odd and two even, it will be found impossible to place them so as to make the sums of all the ranges even.

Every pair of numbers in a range must produce a sum which is repeated one or more times. For instance, $a+b=g+h$, $a+f=p+l$. It follows from this that both 2 and 3 must be out of range with 1, for the remaining 31 and 30 can each be made up only in one way. Similarly 15 and 14 must both be out of range with 16.

Every number must form a part of *six* distinct ranges. Thus k is in *cgkp*, *ijkl*, *afkq*, *acik*, *fgjk*, *klpq*.

Every pair in a range occur again in company in one and only one more range. Thus c, h occur in *cdgh* and *chin*; d, j occur in *dgjm* and *dbjl*.

There are nineteen sets of three numbers which in range with 1 make up 34. From these we have to find six sets such that every number occurs in two and only two sets. From the list must be excluded 16 15 2, 16 14 3 as impossible, and 15 13 5, 15 11, 7, 13 11 9 as being all odd.

First, attempting to proceed without using 16, 15 14 4 is impossible, 15 12 6, 15 10 8 require 14 13 6. This involves 13 12 8, and no second 14 is possible, for 14 12 7 gives 7 only once; 14 11 8 gives 8 three times. Hence 1 must be in range with 16, and any arrangement of the square must be one of the six symmetrical forms.

Every range must consist of two numbers greater than 8, and two not greater than 8. For let $2s = 17$, and let $2t$, $2u$, $2v$, $2w$ represent different odd numbers not greater than 15. Let the top row of the square, if possible, be

$$s-t \quad s-u \quad s-v \quad s+t+u+v,$$

giving three numbers not greater than 8. Suppose that the square falls under the fifth kind of symmetry, where $a+i=17$. Fill up e by $s+w$, and complete the square, when the scheme will stand thus:—

$$\begin{array}{cccc}
s-t & s-u & s-v & s+t+u+v \\
s+w & s+t+u-w & s-t-2u-v+w & s+u+v-w \\
s+t & s+u & s+v & s-t-u-v \\
s-w & s-t-u+w & s+t+2u+v-w & s-u-v+w
\end{array}$$

If

$a=f$, $w=2t+u$; $a=g$, $w=2u+v$; $a=h$, $w=t+u+v$;
$a=p$, $w=2t+2u+v$; $a=q$, $w=u+v-t$;
$b=f$, $w=t+2u$; $b=p$, $w=t+3u+v$; $c=h$, $w=u+2v$;
$c=n$, $w=t+u-v$; $c=p$, $w=t+2u+2v$; $d=g$, $w=2t+3u+2v$;
$e=f$, $2w=t+u$; $e=h$, $2w=u+v$; $e=p$, $2w=t+2u+v$;
$f=g$, $2w=2t+3u+v$; $g=h$, $2w=t+3u+2v$; $g=p$, $w=t+2u+v$;

and the converses of these (which are going to be used) are also true. The other equalities give either manifest impossibilities or useless repetitions. Now $t+u+v$ must be less than s, or else l would be impossible. The only values, therefore, for $2t$, $2u$, $2v$ are:—

1	3	5	leaving	7	9	11	13	15	for $2w$
1	3	7	,,	5	9	11	13	15	,,
1	3	9	,,	5	7	11	13	15	,,
1	3	11	,,	5	7	9	13	15	,,
1	5	7	,,	3	9	11	13	15	,,
1	5	9	,,	3	7	11	13	15	,,
3	5	7	,,	1	9	11	13	15	,,

and every one of these values of $2w$ fulfils one of the equations given above, renders two elements of the square equal, and thus involves an impossibility. The same reasoning applies if any other form of symmetry be chosen, for the same terms will appear in the scheme, only differently placed. Also, if the top row had been $s+t$, $s+u$, &c., or if $s-w$ had been placed at e, the same equations would have reappeared.

The list of numbers making a range with 1 must now be further corrected by striking out 14 10 9, 12 11 10 as consisting of numbers greater than 8, and then striking out 16 12 5, 16 9 8 as containing numbers not repeated. The list then becomes

16	13	4	15	14	4	14	13	6	13	12	8
16	11	6	15	12	6	14	12	7			
16	10	7	15	10	8	14	11	8			

Now for the six ranges including 1 there must be two with 16, and there are three sets which may easily be filled up, thus—

16	13	4	16	13	4	16	11	6
16	11	6	16	10	7	16	10	7
15	14	4	15	14	4	15	12	6
15	12	6	15	10	8	15	10	8
14	11	8	14	12	7	14	12	7
13	12	8	13	12	8	14	11	8

By taking the differences of these from 17, we get the ranges of which 16 is a member, namely—

1	4	13	1	4	13	1	6	11
1	6	11	1	7	10	1	7	10
2	3	13	2	3	13	2	5	11
2	5	11	2	7	9	2	7	9
3	6	9	3	5	10	3	5	10
4	5	9	4	5	9	3	6	9

We are now able to form any square, and to tell the number of arrangements. First, place 1 in any of the 16 positions. Secondly, put 16 in any one of the 9 places in range with 1. Thirdly, choose any one of the three possible pairs of ranges with 1 and 16. Fourthly, place in range the numbers so chosen. This can be done in 8 ways, as will be seen by the example to be given presently. The square can now be completed in only one way, which is determined by considering the necessary ranges of which 16 forms an element. Thus there are $16 \times 9 \times 3 \times 8$, or 3456 variations. But these include repetitions, such as may be made by turning the square round, bringing each side in succession to the top, and by reversing the order of the rows or columns. These two causes bring about eight variations for each distinct set. There are, therefore, 432 essentially different arrangements.

Example.—Place 1 at *j*. Place 16 at *l*, giving the fifth symmetrical system. Chose the 16 13 4, 16 10 7 set of ranges. Then we obtain the *eight* partly formed squares—

.	10	.	7		.	7	.	10		.	10	.	7		.	7	.	10
.	.	.	.		.	.	.	.		.	.	.	.		.	.	.	.
13	1	4	16		13	1	4	16		4	1	13	16		4	1	13	16
.	.	.	.		.	.	.	.		.	.	.	.		.	.	.	.

.	13	.	4		.	4	.	13		.	13	.	4		.	4	.	13
.	.	.	.		.	.	.	.		.	.	.	.		.	.	.	.
10	1	7	16		10	1	7	16		7	1	10	16		7	1	10	16
.	.	.	.		.	.	.	.		.	.	.	.		.	.	.	.

To complete these, remember that 2 must be in range with 16, 7 and 16, 13. This gives as its places in the above eight squares, *h*, *e*, *q*, *p*, *p*, *q*, *e*, *h*. The place of 15 is then determined, and the square easily filled.

There is a simpler practical method of forming the squares, which requires no knowledge of the range numbers, but is not suited to furnish a plan for a demonstration.

RULES.

1. Choose any place for 1.
2. Choose any of the *nine* available places for 16. This determines the nature of the symmetry.
3. Take for 2 any of the six places not in range with 1, provided that the consequent place of 15 is not out of range with 1. There will always be *four* of the six places that will suit, and two that will not.

4. Take for 3 one of the remaining places out of range with 1, and not involving any immediately apparent impossibility. There will always be *three* places suitable. Remember that 2, 3 must not form part of an open range.

5. 4 and 13 will have their places determined by the above operations.

6. There are then always *two* and only two places for 5. These may be more easily found by remembering that 5 cannot go in a range with two numbers making up 14, 15, or 16, as the range could not be completed.

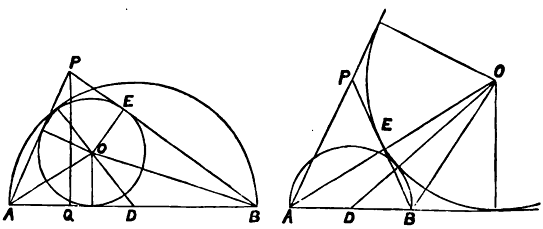
Number of arrangements $16 \times 9 \times 4 \times 3 \times 2 = 3456$ as before.

Question 6502 gives the particular case of the third system of symmetry, $a + k = 16$. There are, therefore, $432 \div 9$ or 48 essentially different arrangements.

6572. (By the EDITOR.)—If from the ends of a diameter AB of a circle, tangents AP, BP be drawn to a circle that touches the given circle and its diameter AB, prove that the sum or the difference of AP and BP will be equal to the sum or the difference of AB and the perpendicular PQ thereon from P, according as the contact of the tangential circle with the given circle and its diameter AB is internal or external.

Solution by W. B. GROVE, B.A.; D. EASTWOOD, M.A.; *and others.*

Let PB meet the tangential circle at E; then $AP \pm BP = AB \pm 2PE$ (according to the figure). We have therefore to prove that $PQ = 2PE$.



Let $AB = c$, radius of tangential circle $(O) = r$; then we have

$(\frac{1}{2}c - r)^2 = OD^2 = r^2 \operatorname{cosec}^2 \frac{1}{2}A + \frac{1}{4}c^2 - cr \cot \frac{1}{2}A$, $\therefore\ \dfrac{r}{c} = \dfrac{1 - \tan \frac{1}{2}A}{\cot \frac{1}{2}A}$.....(1, 2).

But $\dfrac{c}{r} = \cot \frac{1}{2}A + \cot \frac{1}{2}B$, therefore $\dfrac{r}{c} = \tan \frac{1}{2}A \tan \frac{1}{2}B$.........(3, 4);

therefore $\quad \frac{1}{2}PQ = \dfrac{rs}{c} = s \tan \frac{1}{2}A \tan \frac{1}{2}B = s - c = s_3 = PE.$

In the second case, $\frac{1}{2}PQ = s - b = s_2 = PE$. Therefore, &c.

[In the foregoing solution, (2) × (3) gives

$$\tan \tfrac{1}{2}A + \tan \tfrac{1}{2}B \equiv \tan OAB + \tan OBA = 1,$$

a characteristic property of the tangency which is, in fact, needed to reduce (4), wherefrom we obtain (in Fig. 1)

$$1 = \frac{2\sin^2 \frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A} + \&c. = \frac{1-\cos A}{\sin A} + \&c. = \text{cosec}\,A - \cot A + \&c.,$$

or PQ (cosec A + cosec B) = PQ (1 + cot A + cot B), or PA + PB = PQ + AB, which furnishes, in this way, another proof of the theorem.

The locus of O is a parabola, and the locus of P a cubic; for which and other properties see Question 6834].

6253. (By B. WILLIAMSON, F.R.S.)—If two straight lines in a moving plane area always touch the involutes to two circles, prove that any other straight line in the moving area will always touch the involute to a circle.

Solution by W. H. BESANT, M.A., F.R.S.

If α, β, γ be the perpendiculars from any fixed point on the sides a, b, c of the triangle formed by the lines, then $a\alpha + b\beta + c\gamma$ is constant; therefore, since $p + \dfrac{d^2p}{d^2\phi}$ is the radius of curvature of the envelop of a straight line, $a\rho_1 + b\rho_2 + c\rho_3$ is constant. The intrinsic equation of an involute of a circle is of the form $s = c\phi^2 + d\phi + e$, and it follows that, if ρ_1 and ρ_2 are each of the form $c\phi + d$, ρ_3 is of the same form.

The method of this solution, which is due to Mr. FERRERS, is given in my *Notes on Roulettes and Glissettes.*

[Another solution is given in *Reprint*, Vol. XXXIII., p. 67.]

6738. (By Prof. CROFTON, F.R.S.)—Prove that

$$D^n f(xD)\, X = f(xD+n)\, D^n X.$$

Solution by J. J. WALKER, M.A.

Whatever ϕ (D) and χ (D) may be, then, by a Theorem recently communicated by me to the Mathematical Society (Proc. Vol. XII., p. 197), we have

$$f[x\phi(D)]\,\chi(D)\,X = \chi(D) f[x\phi(D)]\,X - \phi(D)\,\frac{d\chi(D)}{dD} f'[\dots]$$
$$+ \frac{1}{1\,.\,2}\left[\phi(D)\frac{d}{dD}\right]^2 \chi(D) f''[\dots]\dots$$

Now, if $\chi(D) = D^n$, $\phi(D) = D$, this gives $f(xD)\,D^n X = D^n f(xD-n)\,X$,

$$\therefore\ f(xD+n)\,D^n X = D^n \{f(xD-n) + nf'(xD-n)$$
$$+ \frac{n^2}{1\,.\,2} f''(xD-n) + \dots\}\,X = D^n f(xD)\,X.$$

[illegible]. (By Professor Cayley, F.R.S.)—Find the stationary and double tangents of the curve $x^4+y^4+z^4=0$.

Solution by the Proposer.

Take i a fourth root of -1, m and n fourth roots of $+1$; then the 28 double tangents are the lines $x=iy$, $x=iz$, $y=iz$, $(4+4+4=)$ 12 lines; and the lines $x+my+nz=0$, 16 lines; and the first 12 of these, each counted twice, are the 24 stationary tangents. In fact, any one of the 12 lines is an undulating tangent, or line meeting the curve in 4 coincident points; it counts therefore once as a double tangent, and twice as a stationary tangent. There should consequently be 16 other double tangents, and it only needs to be shown that these are the 16 lines $x+my+nz=0$. Consider any one line $x+my+nz=0$; for its intersections with the curve $x^4+y^4+z^4=0$, we have

$$(my+nz)^4+y^4+z^4=0,$$

or, as this may be written, $(my+nz)^4+m^4y^4+n^4z^4=0$,

viz., this is $\qquad 2(1,\ 2,\ 3,\ 2,\ 1\between my,\ nz)^4=0,$

or, what is the same thing, $\quad 2\{(1,\ 1,\ 1\between my,\ nz)^2\}^2=0$;

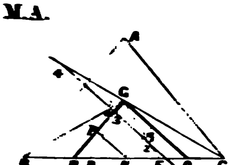
so that the line is a double tangent, the two points of contact being given by means of the equation $(1,\ 1,\ 1\between my,\ nz)^2=0$; viz., ω being an imaginary cube root of unity, we have $nz=\omega\, my$ or $\omega^2 my$; and thence, for the points of contact, $x:y:z=1:\dfrac{\omega}{m}:\dfrac{\omega^2}{n}$, or $=1:\dfrac{\omega^2}{m}:\dfrac{\omega}{n}$;

values which satisfy, as they should do, the two equations

$$x+my+nz=0 \text{ and } x^4+y^4+z^4=0.$$

5527. (By W. S. B. Woolhouse, F.R.A.S.)—Determine, by a simple construction, the size of the equilateral triangle from or into which a triangle can be orthogonally projected.

Solution by C. Leudesdorf, M.A.



Let ABC be a triangle, G its centre of gravity; and let BC be bisected at A and trisected at D and E; join G to B, D, E, C. In order to project ABC into a equilateral triangle, it will be sufficient to project so that BGE, DGC both become right angles. Draw with centres D and E two circles, each of radius DE, and through G and the points where these circles cut, draw a third circle cutting BC in P and Q. Then GP, GQ are the right-angled pair of the involution of which GB, GE; GD, GE are pairs; so that, if a transversal be drawn parallel either to GP or GQ, as 41032 in the figure, the point O will be the centre of the involution 12, 34, and

therefore semicircles described on 12, 34 as diameters, on the side nearer to G, will cut OG in the same point g. Taking g as the orthogonal plane projection of G, and the transversal as the axis, the projection of any point is at once found by drawing an ordinate to the transversal, and diminishing it in the ratio $Og : OG$; and the angles BGE, DGC will both become right angles, and ABC an equilateral triangle. By drawing a transversal parallel to GP, we obtain another projection, which gives another equilateral triangle, that from which ABC can be orthogonally projected. The magnitude of the triangles may be found without actually constructing them. For GP, GQ are the axes of the ellipse touching the sides of ABC at their middle points; so that, if $A'p$, $A'q$ be drawn at right angles to GP, GQ respectively, the squares of the semi-axes are equal to $Gp \cdot GP$ and $Gq \cdot GQ$. If, then, circles be drawn on $A'P$ and $A'Q$ as diameters, the tangents to them from G will be equal in length to the radii of the circles which can be inscribed in the two equilateral triangles.

[Other solutions are given in the *Reprints*, Vol. XXIX., p. 83, and Vol. XXXI., p. 18.]

6411. (By Professor WOLSTENHOLME, M.A.)—If

$$y = x^n - nx^{n-2} + \frac{n(n-3)}{2}x^{n-4} - \frac{n(n-4)(n-5)}{3!}x^{n-6} + \dots,$$

where n is an odd integer $(2r+1)$, prove that $\dfrac{y-2}{x-2} = (u_r + u_{r-1})^2$,

u_r being $$x^r - (r-1)x^{r-2} + \frac{(r-2)(r-3)}{2!}x^{r-4} - \dots .$$

Solution by J. HAMMOND, M.A.; G. HEPPEL, M.A.; *and others.*

Let $x = 2\cos\theta$; then $y = 2\cos n\theta$ and $u_r = \dfrac{\sin(r+1)\theta}{\sin\theta}$,

therefore $$\frac{y-2}{x-2} = \frac{1-\cos n\theta}{1-\cos\theta} = \frac{\sin^2\frac{1}{2}n\theta}{\sin^2\frac{1}{2}\theta},$$

and

$$u_r + u_{r-1} = \frac{\sin(r+1)\theta + \sin r\theta}{\sin\theta} = \frac{2\cos\frac{1}{2}\theta\sin\frac{1}{2}(2r+1)\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \frac{\sin\frac{1}{2}(2r+1)\theta}{\sin\frac{1}{2}\theta},$$

therefore $$\frac{y-2}{x-2} = (u_r + u_{r-1})^2, \text{ if } n = 2r+1.$$

5938. (By W. H. H. HUDSON, M.A.)—A sphere touches another sphere internally, find the centre of gravity of the mass included between them. If the smaller sphere increase in size till it ultimately coincides with the larger, find the final position of the centre of gravity of the

included mass. According to what law of density must matter be distributed over the surface of a sphere that its centre of gravity may coincide with the one just found?

Solution by (1) W. J. C. SHARP, M.A.; (2) *the* PROPOSER.

1. Let r, r' be the radii of outer and inner spheres, and x the distance of centre of gravity, which evidently lies in the diameter through the point of contact, from this point; then

$$r'^3 r' + (r^3 - r'^3)\, x = r^3 r, \quad \text{therefore } x = \frac{r^4 - r'^4}{r^3 - r'^3} = \tfrac{4}{3} r \text{ at limit.}$$

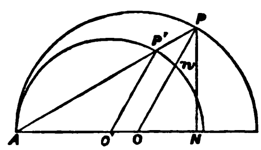
Now, if the masses of the rings be supposed collected at their centres, and $\phi(x)$ be the law of density, we have

$$\frac{\int_0^{2r} \phi(x)\, x\, dx}{\int_0^{2r} \phi(x)\, dx} = \tfrac{4}{3} r, \text{ and if } \phi(x) = \frac{d^2 y}{dx^2} = \psi''(x), \ \frac{2r\psi'(2r) - \psi(2r) + \psi(0)}{\psi'(2r) - \psi'(0)} = \tfrac{4}{3} r;$$

hence, if $\psi(x) = y$, it satisfies the equation; $x\dfrac{dy}{dx} - 3y + 3c + 2c'x = 0$, and $y = c + c'x + c''x^3$, and $\phi(x) = \dfrac{d^2 y}{dx^2} = 6c''x = \mu x$, which is the law of density of the collected masses, whilst that of the surface $= \tfrac{1}{2}\mu x r^{-1}$.

[This analytical solution has the advantage of being not a mere special result, but an example of a method that could be generally used.]

2. The density of the matter distributed on the surface of the sphere will follow the same law as that of the thickness between the two spheres when the inner is on the point of coinciding with the outer.



Let A be the point of contact; O′, O the centres of the inner and outer spheres; P any point on the outer; join AP, OP, meeting the inner sphere in P′, n respectively, and draw PN perpendicular to AO; then A is the centre of similitude, therefore $P'P \propto AP$. Now, ultimately, $Pn = PP' \cos APO$; hence the required law of density $\propto AP \cos PAO \propto AN \propto AP^2$; it is therefore that of the square of the distance from a fixed point on the sphere.

6801. (By C. W. MERRIFIELD, F.R.S.)—Prove that the continued product $m^n!$, where m is prime, is divisible without remainder by m to the power of $(m^n - 1) : (m - 1)$, and the quotient is not again divisible by m.

Solution by W. B. GROVE, B.A.; BELLE EASTON; *and others.*

Consider first $m^3!$; the only factors divisible by m are

$$m,\ 2m,\ 3m,\ \ldots\ m \,.\, m,\ (m+1)\, m,\ \ldots\ 2m \,.\, m,\ (2m+1)\, m,\ \ldots\ 3m^2,\ \ldots\ 4m^2,\ \ldots\ m^3.$$

The number of these factors is m^2; dividing each of these by m, since m is prime, there remain only the quotients from the factors m^2, $2m^2$, ... m^3 still divisible by m; the number of these is m. Dividing these, there remains only one quotient, namely that from m^3, divisible still by m. Therefore $m^3!$ is divisible by m to the power (m^2+m+1) only, that is, $m^3-1 : m-1$. The proof may be easily completed by induction.

[The theorem results from a generalization of the following:—

$$\begin{array}{cccccccccccccccccc}
1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 16 & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 2^3 \\
 & & & 1 & & & & 1 & & & & 1 & & & & 1 & & 2^2 \\
 & & & & & & & 1 & & & & & & & & 1 & & 2 \\
 & & & & & & & & & & & & & & & 1 & & 1
\end{array}$$

the units representing the number of times 2 enters. Summing these by lines, we have $2^3+2^2+2+1 = \dfrac{2^4-1}{2-1}$; and writing m for 2 and n for 4, which is obviously permissible, we get the theorem.]

6597. By Professor Townsend, M.A.)—A uniform flexible cord, in free equilibrium under the action of a central repulsive force varying inversely as the square of the distance, being supposed to have throughout its entire extent the tension to infinity under the action of the force; determine, for any assigned positions of its two terminal points with respect to the centre of force, the requisite length of the cord and the appropriate form of the catenary.

Solution by the Proposer; G. F. Walker, M.A.; *and others.*

The tension throughout the entire extent of a uniform flexible cord, in free equilibrium under the action of any central force, varying in all cases inversely as the perpendicular p upon the tangent from the centre of the force, and varying in the case in question inversely as the distances from the centre, in order to have its tension throughout that to infinity under the action of the force, the form of the catenary is consequently such that throughout its entire extent the ratio of p to r is constant. It is therefore an arc either of a circle or of a logarithmic spiral, according as its terminal radii are equal or unequal; and its length, in either case, being given immediately with the lengths and directions of its terminal radii, therefore &c.

6026. (By W. H. H. Hudson, M.A.)—If P, Q be two points on an equiangular spiral such that the tangents and normals thereat intersect at right angles in T, N respectively, prove that the locus of N is the evolute of the locus of T.

Solution by W. J. C. SHARP, M.A.

Since the angles OQT and OTQ are supplementary, a circle can be described through OPTQN, the centre C of which is the intersection of PQ and TN. Then OPT is a constant angle, and so is OQP = OTP, and consequently TOP, hence the locus of T is an equiangular spiral; and since ONT = OQT a constant, OT : ON is constant; and since TON is a right angle, the locus of N is also an equiangular spiral, which is the evolute of the locus of T. (SALMON's *Higher Plane Curves*, p. 281.)

6780. (By R. RAWSON.)—BOOLE has given (*Diff. Eqs.*, p. 459) the theorem from CURTIS (*Camb. Math. Jour.*, Vol. ix., p. 280)

$$\frac{d^2u}{dx^2}+2Q\frac{du}{dx}+\left\{Q^2+\frac{dQ}{dx}\pm c^2-\frac{m(m+1)}{x^2}\right\}u=0,$$

which can be integrated in finite terms when Q is any function of x. Show that the more general equation

$$\frac{d^2u}{dx^2}+2Q\frac{du}{dx}+\left\{Q^2+\frac{dQ}{dx}+Ax^r+\frac{(m-r)(m+r+4)}{4(m+2)^2x^2}\right\}u=0$$

can be integrated in finite terms, where A is any constant, and $(2p+1)m=-4p$; p being any integer.

I. *Solution by* ARTHUR HILL CURTIS, LL.D., D.Sc.

The given equation is at once reducible to

$$\left\{\left(\frac{d}{dx}+Q\right)^2+Ax^r+\frac{(m-r)(m+r+4)}{4(m+2)^2x^2}\right\}u=0,$$

or, putting $\int Q\,dx=Q_1$, to

$$\left\{\frac{d^2}{dx^2}+Ax^r+\frac{(m-r)(m+r+4)}{4(m+2)^2x^2}\right\}e^{Q_1}u=0;$$

and, putting $e^{Q_1}u=v$, we get

$$\left\{\frac{d^2}{dx^2}+Ax^r+\frac{(m-r)(m+r+4)}{4(m+2)^2x^2}\right\}v=0,$$

or
$$\left\{x^2\frac{d^2}{dx^2}+Ax^{r+2}+\frac{(m-r)(m+r+4)}{4(m+2)^2}\right\}v=0.$$

Let $x=e^\theta$, and let $\frac{d}{d\theta}$ be denoted by D; then the equation is reducible to

$$\left\{D(D-1)+\frac{(m-r)(m+r+4)}{4(m+2)^2}+Ae^{(r+2)\theta}\right\}v=0,$$

or
$$\left\{\left(D-\frac{m-r}{2(m+2)}\right)\left(D-\frac{m+r+4}{2(m+2)}\right)+Ae^{(r+2)\theta}\right\}v=0,$$

or, putting $v=ze^{\frac{m-r}{2(m+2)}\theta}$, $\left\{D\left(D-\frac{r+2}{m+2}\right)+Ae^{(r+2)\theta}\right\}z=0.$

Put $(r+2)\theta = (m+2)\phi$, then $D = \frac{d}{d\theta} = \frac{(r+2)}{(m+2)}\frac{d}{d\phi}$,
and the equation becomes

$$\left\{\left(\frac{r+2}{m+2}\right)^2 \frac{d}{d\phi}\left(\frac{d}{d\phi}-1\right) + Ae^{(m+2)\phi}\right\} z = 0;$$

or, putting $e^\phi = y$, $\left\{y\frac{d}{dy}\left(y\frac{d}{dy}-1\right) + A\left(\frac{m+2}{r+2}\right)^2 y^{m+2}\right\} z = 0,$

or $$\left\{y^2\frac{d^2}{dy^2} + A\left(\frac{m+2}{r+2}\right)^2 y^{m+2}\right\} z = 0,$$

or $$\frac{d^2z}{dy^2} + A\left(\frac{m+2}{r+2}\right)^2 y^m z = 0,$$

and, if $m = -\frac{4p}{2p+1}$, this equation is identical with RICCATI's equation, which, as is well known, can be solved, when p is an integer, in a variety of ways. One method is given in my paper referred to by Mr. RAWSON.

When z has been found as a function of y, it is known as a function of x by the relation $y = e^\phi = (e^\theta)^{\frac{r+2}{m+2}} = x^{\frac{r+2}{m+2}}$, and u is known by the relation $u = e^{-Q_1}v = e^{-Q_1}x^{\frac{m-r}{2(m+2)}}z$.

II. *Solution by the* PROPOSER.

The relations $$x_1 = x^n, \quad y_1 = \frac{a_1 y}{anx^{n-1}} + \frac{n-1}{2anx^n} \quad \text{.....................(1)}$$

transform $\frac{dy_1}{dx_1} + ay_1^2 = bx_1^m + \frac{c}{x_1^2}$ into $\frac{dy}{dx} + a_1y^2 = Ax^r + \frac{c_1}{x^2}$......(2, 3),

where $A = \frac{abn^2}{a_1}$, $c_1 = \frac{(4ac+1)n^2-1}{4a_1}$, $r = (m+2)n-2$.

If $c = 0$, and m satisfies RICCATI's condition of integrability, then equation (3) is soluble by substituting in (1) the value of y_1, obtained by RICCATI's process in (2).

Then $$c_1 = \frac{n^2-1}{4a_1} = \frac{(r+2)^2-(m+2)^2}{4a_1(m+2)^2} = \frac{(r-m)(r+m+4)}{4a_1(m+2)^2};$$

and, when $a_1 = -1$, (3) becomes

$$\frac{dy}{dx} - y^2 = Ax^r + \frac{(m-r)(m+r+4)}{4(m+2)^2x^2} \quad \text{.....................(4).}$$

RICCATI's condition is satisfied by $(2p+1)m + 4p = 0$, giving for the value of n the quantity $\frac{1}{2}(r+2)(2p+1)$.

Again, let $\frac{dy}{dx} + (Q-v)y = 0$, $\frac{du}{dx} + (Q+v)w = y$ ....(5, 6);

therefore $\frac{d}{dx}\left\{\frac{du}{dx} + (Q+v)w\right\} + (Q-v)\left\{\frac{du}{dx} + (Q+v)w\right\} = 0,$

or $$\frac{d^2u}{dx^2} + 2Q\frac{du}{dx} + \left\{Q^2 + \frac{dQ}{dx} + \frac{dv}{dx} - v^2\right\}u = 0 \quad \text{.................(7).}$$

If, then, v is determined from

$$\frac{dv}{dx} - v^2 = Ax^r + \frac{(m-r)(m+r+4)}{4(m+2)^2 x^2}$$

by means of (4), then the equation in the Question is soluble by means of (5) and (6). The values $r = 0$ and $A = \pm c^2$ give CURTIS's theorem.

6446. (By Professor GENESE, M.A.)—If $u \equiv (a, b, c, f, g, h)(x, y, 1)^2 = 0$ be the equation to a conic, prove that (1) the *directrices* are determined by the equation $\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4\lambda u$, where λ is the root of the equation $\lambda^2 - (a+b)\lambda + ab - h^2$; and (2) that, if α, β be the coordinates of the centre, the equation to the axes is

$$\frac{du}{dx} : \frac{du}{dy} = x - \alpha : y - \beta.$$

Solution by Professor WOLSTENHOLME, M.A.

If $u \equiv (a, b, c, f, g, h)(x, y, 1)^2$ be the equation of a conic referred to rectangular coordinates, the equation of its director circle is

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4(a+b)u.$$

Now the two directrices (real or impossible) of a conic pass through the four common points of a conic and its director circle, as is obvious by taking the forms $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $x^2 + y^2 = a^2 + b^2$, which give $x^2 = \frac{a^4}{a^2 - b^2}$ taken simultaneously. Hence the equation of a pair of directrices will be of the form $\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4\lambda u$, λ being sufficiently determined by making the discriminant of this equation vanish. This gives the cubic

$$\begin{vmatrix} a^2 + h^2 - \lambda a, & h(a+b-\lambda), & ag + hf - \lambda g \\ h(a+b-\lambda), & b^2 + h^2 - \lambda b, & bf + hg - \lambda f \\ ag + hf - \lambda g, & bf + hg - \lambda f, & f^2 + g^2 - \lambda c \end{vmatrix} = 0;$$

but [as shown in the solution to Quest. 6038; *Reprint*, Vol. XXXII., p. 91], this equation reduces to $\Delta\lambda\,[(\lambda - a)(\lambda - b) - h^2] = 0$ [of which the significant factor might be obtained at once by making the equation parabolic].

The root $\lambda = 0$ gives us the two diameters through the four common points, viz., $\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 0$; hence the equation of the two pairs of directrices is $\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4\lambda u$, where λ may have either value given by the equation $(\lambda - a)(\lambda - b) = h^2$, one root giving the

real directrices and the other the impossible. If $ab = h^2$ another root of the equation is zero, and we get the unique directrix $\left(\frac{du}{d}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4(a+b)u$, the terms of two dimensions disappearing. I arrived at the above result for the determinant in the investigation of this same equation. When I set the question in an examination here, I gave an example to illustrate the fact that, although this is the simplest form of general equation of directrices, it is by no means a convenient one to use in practice if the numerical coefficients are at all large. To get the *real* directrices, the greater algebraical root should be taken, if Δ, *i.e.*, the discriminant $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$, be negative. In oblique coordinates, the equations will be

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 - 2\frac{du}{dx}\frac{du}{dy}\cos\omega = 4\lambda u,$$

$$\lambda^2 - \lambda(a+b-2h\cos\omega) + (ab-h^2)\sin^2\omega = 0.$$

As an example (a very favourable one since there are no surds involved) take $\quad u = 7x^2 + 23y^2 + 293 - 134y - 52x - 12xy$;

then $\quad \Delta = 7\times 23\times 293 - 2\times 67\times 26\times 6 - 7\times 67^2 - 23\times 26^2 - 293\times 6^2$

$$= 293\times 125 - 67\times(312+469) - 23\times 676 = -37950;$$

the roots of the equation $(\lambda - a)(\lambda - b) = h^2$ are 5 and 25; hence the equation of the real directrices is

$$(7x-6y-26)^2 + (-6x+23y-67)^2 = 25(7x^2+23y^2+\ldots),$$

or $$90x^2 + 60xy + 10y^2 - 580(3x+y) + 2160 = 0,$$

or $\quad (3x+y)^2 - 58(3x+y) + 216 = 0,\quad$ therefore $3x+y = 4$ or 54.

For the foci, $\dfrac{7x-6y-26}{3} = \dfrac{-6x+23y-67}{1} = \dfrac{26x+67y-293}{4 \text{ or } 54}$,

whence the major axis is $x - 3y + 7 = 0$; and we have, on substituting for x

$$5y - 25 = \frac{135y-475}{4 \text{ or } 45},$$

(1) $4y - 20 = 29y - 95,\quad y = 3,\quad$ and therefore $x = 2$.

(2) $54y - 270 = 29y - 95,\quad y = 7,\quad$ and $x = 14$.

In reducing, all the coefficients need not be calculated (unless as a test of accuracy) since we know that the equation represents two *parallel* straight lines; it will be sufficient to find those of x^2, xy, x, and the constant term.

6722. (By D. EDWARDES.)—Prove that the mean distance from its centre of all points within an oblate spheroid, of equatorial radius a, is

$$\tfrac{3}{8}(ae^{-1}\sin^{-1}e + b).$$

Solution by Rev. T. R. Terry, M.A., F.R.A.S.; Kate Gale; *and others.*

$$\text{Mean distance} = \frac{3}{4\pi a^2 b} 2 \iint dx\, dy\, 2\pi x\, (x^2+y^2)^{\frac{1}{2}} \text{ (over } \tfrac{1}{4} \text{ of ellipse)}$$

$$= \frac{1}{a^2 b} \int_0^b \left\{ \frac{a^3}{b^3} (b^2 - e^2 y^2)^{\frac{3}{2}} - y^3 \right\} dy = \frac{a}{e} \int_0^{\sin^{-1} e} \cos^4 \theta \, d\theta - \frac{b^3}{4a^3},$$

where $ey = b \sin \theta = \frac{3}{8} (ae^{-1} \sin^{-1} e + b)$.

6574. (By W. B. Grove, B.A.)—Suppose a series of 73 cards to be painted with red, blue, yellow, and green, every card but one receiving at least one colour. Let it be observed that 21 have *some* part coloured red, 48 blue, 31 yellow, and 46 green; also 14 have both red and blue upon them, 16 both red and green, 14 both blue and yellow, 28 both blue and green, 20 both yellow and green, and 9 have all four colours. Also 15 are painted with blue *alone*, 6 with yellow alone, but none with either red or green alone. Find the laws (designed or accidental) according to which the colours are arranged.

Solution by Hugh McColl, B.A.

Taking any card out of the series, and speaking of this card throughout, let r, b, y, g respectively denote the statements, it is *red*, it is *blue*, it is *yellow*, it is *green*; and let ϵ (an equivalent symbol for 1) denote the admitted statement, it is *one of the given series*. Also, let a_n denote the number of cards in the series to which any statement a is applicable, so that $a_n = 73a_c$, in which a_c denotes the chance that a is true.

Then, our data are: $r_n = 21$, $b_n = 48$, $y_n = 31$, $g_n = 46$, $(r+b+y+g)_n = 72$, $(rb)_n = 14$, $(rg)_n = 16$, $(by)_n = 14$, $(bg)_n = 28$, $(yg)_n = 20$, $(rbyg)_n = 9$, $(r'by'g')_n = 15$, $(r'b'yg')_n = 6$, $(rb'y'g')_n = 0$, $(r'b'y'g)_n = 0$. Applying rules 2 and 4 of my fourth paper in the *Proceedings* of the London Mathematical Society, we readily get the 5 equations

$$(rby)_n + (rbg)_n + (ryg)_n + (byg)_n - (ry)_n = 27, \quad (rby)_n + (rbg)_n + (byg)_n = 32,$$

$$(rby)_n + (ryg)_n + (byg)_n - (ry)_n = 18, \quad (rby)_n + (ryg)_n + (rbg)_n - (ry)_n = 18,$$

$$(byg)_n + (ryg)_n + (rbg)_n = 27;$$

from which we get $(rby)_n = 14$, $(rbg)_n = 9$, $(ryg)_n = 9$, $(byg)_n = 9$, $(ry)_n = 14$. Applying rule 4 (when necessary) to determine the number of cards to which any four-factor statement (such as $rb'y'g$) is applicable, we get $(rbyg)_n = 9$, $(rbyg')_n = 5$, $(rb'y'g)_n = 7$, $(r'by'g)_n = 19$, $(r'by'g')_n = 15$, $(r'b'yg)_n = 11$, $(r'b'yg')_n = 6$, $(r'b'y'g')_n = 1$. The remaining 8 compound statements are not applicable to any card, and, reducing their sum to its primitive form, we get $r(b'y + by' + b'g') + r'(by + b'y'g) : 0$, which is equivalent to the equation $r = by + b'y'g$.

6667. (By the EDITOR.)—Given the distances of a point from three of the corners of a square, (1) construct the square; and prove (2) that if a, b, c be these distances and Δ the area of a triangle whose sides are a, $b\surd 2$, c, the area of the square is $\frac{1}{2}(a^2+c^2)\pm 2\Delta$; also (3) extend the problem to the case of any rectangle or triangle given in species, showing that, in the case of a rectangle whose sides are as $m : n$ and area Σ, if Δ be the area of a triangle with sides a, b', c', where

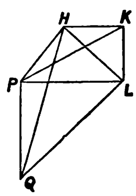
$$b' = \frac{(m^2+n^2)^{\frac{1}{2}}}{n}\,b,\quad c' = \frac{m}{n}\,c,\quad \text{then } \Sigma = \frac{1}{m^2+n^2}\left\{mn\,(a^2+c^2) \pm 4n^2\Delta\right\},$$

$+$ or $-$ according as the angle subtended at the point by the diagonal is obtuse or acute.

I. *Solution by* G. HEPPEL, M.A.; LIZZIE A. KITTUDGE; *and others.*

1. Let A, B, C be three consecutive corners of any square. Divide AB externally and internally in the ratio of $a : b$ by the points D, E. On DE draw a semicircle, which will be the locus of points whose distances from A and B are as $a : b$. Determine a similar locus for BC and $b : c$. These circles will, in general, cut in two points. If Q be either of these, at the given point P make angles HPK, KPL equal to AQB, BQC, and take PH $= a$, PK $= b$, PL $= c$; HKL will be the required square.

2. The construction for a rectangle or triangle given in species is precisely similar, starting with any rectangle or triangle of the given shape.



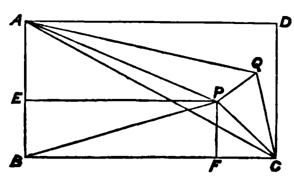
3. Draw PQ at right angles and equal to PL. Join HL, HQ, LQ. Then $\angle$ KLP = HLQ, and HL = KL$\surd 2$, QL = PL$\surd 2$; hence HQ = PK$\surd 2$. Now

$$\text{HL}^2 = \text{twice } \square = a^2+c^2-2ac\cos\text{HPL}$$
$$= a^2+c^2-2ac\sin\text{HPQ} = a^2+c^2-4\Delta\text{HPQ},$$

therefore $\square = \frac{1}{2}(a^2+c^2)-2\Delta$.

In the case where the angle HPL is obtuse, we shall find that $\square = \frac{1}{2}(a^2+c^2)+2\Delta$.

II. *Solution by the* PROPOSER.



1. Taking the general case of a right-angled triangle ABC and rectangle ABCD (the case for *any* triangle being precisely similar), draw PE, PF parallel to the sides, and put

$$\text{PA} = a,\ \text{PB} = b,\ \text{PC} = c,$$
$$\text{AB} = mx,\ \text{BC} = nx;$$

then we have

$$\text{BE} = \frac{b^2+m^2x^2-a^2}{2mx},\quad \text{BF} = \frac{b^2+n^2x^2-c^2}{2nx};\ \text{and } \text{BE}^2+\text{BF}^2 = \text{BP}^2;$$

whence $n^2\left\{m^2x^2+(b^2-a^2)\right\}^2+m^2\left\{n^2x^2+(b^2-c^2)\right\}^2 = 4m^2n^2b^2;$

$$m^2n^2(m^2+n^2)\,x^4-2m^2n^2(a^2+c^2)\,x^2+\left\{m^2(b^2-c^2)^2+n^2(b^2-a^2)^2\right\} = 0;$$

$$\frac{m^2+n^2}{a^2+c^2}x^2 = 1\pm\left\{1-\frac{(1+m^2n^{-2})(b^2-c^2)^2+(1+n^2m^{-2})(b^2-a^2)^2}{(a^2+c^2)^2}\right\}^{\frac{1}{2}};$$

$$(m^2+n^2)\,x^2 = a^2+c^2 \pm \left\{ (a^2+c^2)^2 - \frac{m^2+n^2}{n^2}(b^2-c^2)^2 - \frac{m^2+n^2}{m^2}(b^2-a^2)^2 \right\}^{\frac{1}{2}}$$

$$= a^2+c^2 \pm \frac{n}{m}\left\{2\,(b'^2c'^2+c'^2a^2+a^2b'^2)-(a^4+b'^4+c'^4)\right\}^{\frac{1}{2}}$$

$$= a^2+c^2 \pm \frac{n}{m}\,.\,4\Delta\,;$$

therefore $\Sigma = mnx^2 = \dfrac{1}{m^2+n^2}\left\{mn\,(a^2+c^2) \pm 4n^2\Delta\right\}$.

2. *Otherwise:* draw PQ perpendicular to PC, and make $\angle$ QCP = ACB; then, in the triangles ACQ, BCP, we have

$$\text{AC} = (1+m^2n^{-2})^{\frac{1}{2}}\,\text{BC},\quad \text{CQ} = (1+m^2n^{-2})^{\frac{1}{2}}\,\text{CP},\quad \angle\,\text{ACQ} = \text{BCP},$$

therefore $\quad \text{AQ} = (1+m^2n^{-2})^{\frac{1}{2}}\,b = b'$; also $\text{PQ} = c'$;

hence $\quad \text{AC}^2 = \text{AP}^2+\text{PC}^2-2\text{AP}\,.\,\text{PC}\cos\text{APC}$

$$= a^2+c^2+2\text{AP}\,.\,\frac{n}{m}\,\text{PQ}\sin\text{APQ} = a^2+c^2+\frac{n}{m}\,.\,4\Delta\,;$$

but $\quad \text{AC}^2 = (m^2+n^2)\,x^2$, and $\Sigma = mnx^2 = \dfrac{mn}{m^2+n^2}\,.\,\text{AC}^2$;

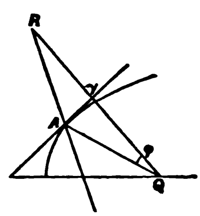
therefore $\Sigma = \dfrac{mn}{m^2+n^2}(a^2+c^2) + \dfrac{4n^2}{m^2+n^2}\Delta = \dfrac{1}{m^2+n^2}\left\{mn\,(a^2+c^2)+4n^2\Delta\right\}$.

3. In the case of a square, we have $m = n$, and then $\Sigma = \frac{1}{2}(a^2+c^2) \pm 2\Delta$.

6612. (By E. B. Elliott, M.A.)—A spherical surface of radius a, made of thin perfectly flexible material whose internal side is reflecting, is deformed without either crumpling or stretching, and assumes the form of a surface of revolution. From a point Q on the axis of this surface, and within it, a small pencil of light is incident on it at any point A. Show that, if q_1, q_2 be the primary and secondary foci of the reflected pencil, and if R be the image of Q by reflection in the tangent plane at A,

$$\text{R}q_1\,.\,\text{R}q_2 : \text{A}q_1\,.\,\text{A}q_2 = 4\text{QA}^2 : a^2.$$

Solution by G. F. Walker, M.A.; Lizzie A. Kittudge; *and others.*



We have $\quad \dfrac{1}{v_1}+\dfrac{1}{u} = \dfrac{2}{\rho_1\cos\phi}$,

and $\quad \dfrac{1}{v_2}+\dfrac{1}{u} = \dfrac{2\cos\phi}{\rho_2}$,

or $\quad \dfrac{\text{R}q_1}{\text{A}q_1\,.\,\text{AQ}} = \dfrac{2}{\rho_1\cos\phi},\quad \dfrac{\text{R}q_2}{\text{A}q_2\,.\,\text{AQ}} = \dfrac{2\cos\phi}{\rho_2}$,

therefore $\quad \dfrac{\text{R}q_1\,.\,\text{R}q_2}{\text{A}q_1\,.\,\text{A}q_2\,.\,\text{AQ}^2} = \dfrac{4}{\rho_1\rho_2} = \dfrac{4}{a^2}$

by the given condition. Therefore

$$\text{R}q_1\,.\,\text{R}q_2 : \text{A}q_1\,.\,\text{A}q_2 = 4\text{AQ}^2 : a^2.$$

6257. (By A. BUCHHEIM, Ph.D.)—Show that

$$\iiint \frac{\sum_1^3 x_i^6 + 3\sum_1^3\sum_1^3 x_i^2 x_j^4 + 2x_1^2 x_2^2 x_3^2}{(\sum_1^3 x_i^2)^6} dx_1 dx_2 dx_3 = \frac{4\pi}{3abc};$$

the integration extending over the whole space outside the pedal of the ellipsoid of semiaxes a, b, c with regard to its centre the origin of co-ordinates.

Solution by the PROPOSER; Professor EVANS, M.A.; *and others.*

The coefficient of $dx_1 dx_2 dx_3$ is the Jacobian of y_1, y_2, y_3 with regard to x_1, x_2, x_3, the points x, y being inverse points with regard to the sphere of radius unity described about the origin; hence the integral gives ths volume of the inverse of the pedal of the ellipsoid, that is, of the ellipsoid $a^2x_1^2 + b^2x_2^2 + c_2x_3^2 = 1$; but this is $\frac{4\pi}{3abc}$.

6529. (By E. W. SYMONS, M.A.)—From a point four normals are drawn to a conic; prove that, if the line joining two of their feet move parallel to itself, the line joining the other two will move parallel to itself; and find the locus of the point in order that this may be possible.

Solution by G. F. WALKER, M.A.; CHRISTINE LADD; *and others.*

The normals at the ends of the chords (1) meet in the point (2),

$$\frac{lx}{a} + \frac{my}{b} = 1,\ \frac{x}{la} + \frac{y}{mb} + 1 = 0 \ldots (1),\ \frac{a^2-b^2}{l^2+m} = \frac{by}{m(l^2-1)} = \frac{ax}{-l(m^2-1)} \ldots (2),$$

and if one be fixed in direction, so is the other.

If we take $l \sec \alpha = m \operatorname{cosec} \alpha = \lambda$, so that the chord is parallel to the tangent at the point α, we get, to find (x, y),

$$\frac{a^2-b^2}{\lambda} = \frac{by}{\sin\alpha(\lambda^2\cos^2\alpha - 1)} = \frac{ax}{-\cos\alpha(\lambda^2\sin^2\alpha-1)} = \frac{ax\cos\alpha + by\sin\alpha}{\cos 2\alpha}$$

$$= \frac{ax\sec\alpha + by\operatorname{cosec}\alpha}{\lambda^2\cos 2\alpha},$$ and the locus of (x, y) is

$$(ax\cos\alpha + by\sin\alpha)\left(\frac{ax}{\cos\alpha} + \frac{by}{\sin\alpha}\right) = (a^2-b^2)^2\cos^2 2\alpha.$$

6431. (By E. W. SYMONS, M.A.)—Prove that the locus of a point, the normals from which to a given ellipse form a harmonic pencil, is

$$(a^2x^2 + b^2y^2 - c^4)^3 + 54a^2b^2 \cdot c^4x^2y^2 = 0 \quad (c^2 \equiv a^2 - b^2).$$

Solution by the PROPOSER; C. H. SWIFT, B.A.; *and others.*

The equation of a normal may be written

$$y = mx - mc^2(a^2 + m^2b^2) \quad ,$$

whence $\quad m^4b^2x^2 - 2m^3b^2xy + m^2(a^2x^2 + b^2y^2 - c^4) - 2a^2mxy + a^2y^2 = 0$

is the relation connecting the angles of inclination to the x-axis of the four normals from (x, y). Now the condition for an harmonic pencil is the invariant $\mathrm{T} = 0$ (see SALMON's *Higher Algebra*, p. 187, art. 206). If $\Sigma \equiv a^2x^2 + b^2y^2 - c^4$, this gives, for locus of point,

$$\tfrac{1}{6}a^2b^2x^2y^2\Sigma + \tfrac{1}{12}a^2b^2x^2y^2\Sigma - \tfrac{1}{4}a^4b^2x^4y^2 - \tfrac{1}{4}a^2b^4x^2y^4 - \tfrac{1}{216}\Sigma^3 = 0,$$

$$a^2b^2x^2y^2\left\{\tfrac{1}{6}\Sigma + \tfrac{1}{12}\Sigma - \tfrac{1}{4}\Sigma - \tfrac{1}{4}c^4\right\} - \tfrac{1}{216}\Sigma^3 = 0,$$

or

$$54a^2b^2c^4x^2y^2 + \Sigma^3 = 0.$$

6555. (By H. STEWART, B.A.)—If

$$\frac{bc(y+z) - a^2x}{a(b^2 + c^2 - bc)} = \frac{ca(z+x) - b^2y}{b(c^2 + a^2 - ca)} = \frac{ab(x+y) - c^2z}{c(a^2 + b^2 - ab)},$$

prove that $(a^2x + b^2y + c^2z)(b^2c^2 + c^2a^2 + a^2b^2) = 3a^2b^2c^2(x + y + z)$.

Solution by the Rev. J. L. KITCHIN, M.A.; J. O'REGAN; *and others.*

Putting each of the fractions $= \lambda$, and solving, we easily obtain

$$x = \lambda\frac{bc}{a}, \quad y = \lambda\frac{ac}{b}, \quad z = \lambda\frac{ab}{c},$$

therefore $a^2x + b^2y + c^2z = 3\lambda abc$; also $x + y + z = \lambda\left(\frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c}\right)$,

hence $abc(x + y + z) = \lambda(b^2c^2 + a^2c^2 + a^2b^2)$, therefore &c.

6465. (By E. W. SYMONS, M.A.)—Prove that the line joining the point where $ax^2 + 2hxy + by^2 + 2gx + 2fy = 0$ is cut by $a'x^2 + 2h'xy + b'y^2 = 0$ is

$$\frac{(\lambda a' - a)}{g}x + \frac{(\lambda b' - b)}{f}y = 2, \text{ wherein } \lambda \equiv \frac{af^2 + bg^2 - 2fgh}{a'f^2 + b'g^2 - 2fgh'}.$$

Solution by R. E. RILEY, M.A.; D. EDWARDES; *and others.*

Assume the required equation to be $lx + my = 1$, then

$$ax^2 + 2hxy + by^2 + 2(gx + fy)(lx + my) = 0$$

is identic with $a'x^2 + 2h'xy + b'y^2 = 0$. Hence we get two linear equations in l, m, which give the result stated in the Question.

6564. (By Professor TOWNSEND, F.R.S.)—A point, taken arbitrarily in the plane of a uniform circular lamina attracting inversely as the square of the distance, and its reflexion with respect to the centre, being supposed the foci, if internal of an ellipse, and if external of an hyperbola, having the containing diameter for transverse axis of figure; show that the potential of the attraction for either point is equal to the mass per unit of area of the lamina, multiplied into the circumference of the ellipse in the former case, and into the difference between the circumference of the hyperbola and the sum of its asymptotes in the latter case.

Solution by the PROPOSER.

Denoting by a the radius of the lamina, by m its mass per unit of area, by c the distance of the point from its centre, by l and θ the length and inclination to the containing diameter of any chord passing through the point, and by V the potential of the attraction, distinguished as V_i or V_e according as the point is internal or external to the mass of the lamina; then, since manifestly, in either case, $dV = ml d\theta$, and $l = 2(a^2 - c^2 \sin^2\theta)^{\frac{1}{2}}$, therefore, at once, in the former case $V_i = 4m \int_0^{\frac{1}{2}\pi} (a^2 - c^2\sin^2\theta)^{\frac{1}{2}} d\theta$, and in the latter case $V_e = 4m \int_0^{\sin^{-1}\frac{a}{c}} (a^2 - c^2 \sin^2\theta)^{\frac{1}{2}} d\theta$; and therefore, &c.

6395. (By E. W. SYMONS, M.A.)—Three normals being drawn to an ellipse from a point in its evolute; prove (1) that the locus of the centre of the circle passing through their feet is $4(a^2x^2 + b^2y^2)^3 = a^2b^2(a^2-b^2)^2 x^2y^2$; and (2) that one of the common chords of the ellipse and circle passes through the centre of the ellipse.

Solution by G. F. WALKER, M.A.; *the* PROPOSER; *and others.*

It is not hard to show that the normals at the points where

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1 = 0 \text{ and } \frac{x}{a\cos\phi} + \frac{y}{b\sin\phi} + 1 = 0$$

meet the ellipse, pass through a point on the evolute.

Thus $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \lambda\left(\frac{x}{a\cos\phi} + \frac{y}{b\sin\phi} + 1\right)\alpha$$

must be a circle passing through $(a\cos\phi, b\sin\phi)$, therefore

$$\alpha \equiv bx\sin\phi - ay\cos\phi;$$

and this being one of the common chords of ellipse and circle, and passing through the centre, proves (2).

Again, expressing that coefficient of x^2 = that of y^2, we get

$$\lambda \equiv \frac{(a^2-b^2)\sin\phi\cos\phi}{ab(a^2\cos^2\phi + b^2\sin^2\phi)},$$ and the equation of the circle is

$$x^2+y^2+\frac{(a^2-b^2)}{a}\sin^2\phi\cos\phi\, x+\frac{(b^2-a^2)}{b}\sin\phi\cos^2\phi\, y-a^2\cos^2\phi-b^2\sin^2\phi=0;$$

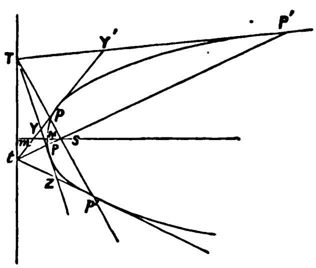
therefore, if (x, y) be the centre, we have

$$2ax = -(a^2-b^2)\sin^2\phi\cos\phi,\quad 2by = (a^2-b^2)\sin\phi\cos^2\phi;$$

and therefore, eliminating ϕ, we get the required result.

6811. (By the Rev. C. TAYLOR, D.D.)—Focal chords of a parabola at right angles to one another meet the directrix in T, t. Show that (1) the bisectors of the angles between the tangents from either of the points T, t are parallel to the tangents from the other; and (2) every pair of the four tangents intersect at constant angles.

Solution by F. BUDD, M.A.; KATE GALE; *and others.*



Since the tangents at P and P' intersect at T at right angles,

$\angle$ Y'TS = SPT

$= \angle$ SMP $= \angle$ YMm,

and $\angle$ Y'pT $= \angle$ Spm $= \angle$ Smp;

$\therefore$ $\angle$ TY'Y $= \angle$ mYM $= \angle$ TYY'

$= \frac{1}{4}\pi =$ PZt, &c.

Therefore the bisector of $\angle$PTP' is at right angles to YY', and therefore parallel to tp', &c.

AN ANALYSIS OF RELATIONSHIP.

By ALEXANDER MACFARLANE, M.A., D.Sc., F.R.S.E.

In this article I propose to give an abstract of several papers on an Analysis of Relationship, which I have recently contributed to the Royal Society of Edinburgh (Proceedings for May 1879, Dec. 1880, and March 1881). The method is a development in a special direction of the principles of the *Algebra of Logic*, and as finally developed enables one to solve problems about relationships by purely analytical processes.

1. The subject considered is, in its widest extent, mankind; in a particular investigation, it may be any specified portion of that natural class. Let U denote any man in the community of men considered, U_A the man whose name is A, U_B the man whose name is B,—the latter may be written A, B, the U being left to be understood as being constant throughout the given investigation.

2. Let cA denote a child of A; then $c^{-1}A$ will denote a parent of A, ccA a grandchild of A, $cc^{-1}A$ a child of a parent of A, $c^{-1}cA$ a parent of a child of A, $c^{-1}c^{-1}A$ a grandparent of A. These expressions can be denoted without ambiguity by c^2A, $c^{1-1}A$, $c^{-1+1}A$, $c^{-2}A$ respectively. The index in c^{1-1} may or may not reduce to 0; if it is irreducible, the relationship means *brother or sister*; if it reduces, the meaning is *self.*

3. The symbol Σ is used to denote *all*, as ΣcA all the children of A. If the value of "the all" is specified, then Σ is replaced by a number with a dot over it, as $\dot{3}cA$, the three children of A. Numbers are also required to express a *definite some*, as $2cA$ two children of A. The former may be called *complete*, the latter *partial* numbers. Numbers are also required in their ordinary sense of coefficients, in which case they may be distinguished by having a larger size as $\mathbf{2}\Sigma cA$ twice the children of A. Subscript numbers as in c_1A, c_2A express the eldest child of A, the second child of A, etc.

4. In such an expression as $(3+2)\,cA$, where two partial numbers are connected by $+$, it is not necessary that each of the individuals counted in the $3cA$ should be different from each of the individuals counted in the $2cA$. The expression $(3+2)\,cA$ is equivalent either to $5cA$ or $(3+2.1)\,cA$ (where the dot marks off the coefficient) or $(1+2.2)\,cA$, where the terms connected are now exclusive of one another. Similarly $(3-2)\,cA$ is not necessarily equivalent to $1cA$. It may be irreducible, or it may reduce to $(2-1)\,cA$, or it may reduce further to cA. Such an expression as $(\dot{3}+2)\,cA$ is equivalent to $(1+2\cdot2)\,cA$, and $(\dot{3}-2)\,cA$ to cA.

5. The symbol Σ placed before a sum of terms is distributive; a definite or partial number is non-distributive.

6. Let the order of a relationship refer to the number of times c enters into its expression, whether directly or inversely; then the general relationships of the n^{th} order are obtained by expanding $(c+c^{-1})^n$, provided the consecutiveness of the symbols in the products be preserved.

For example,

Term.	General Meaning.	Meaning, if irreducible.
c^3	Great-grandchild	
c^{2-1}	Grandchild of parent	Nephew or niece
c^{1-1+1}	Child of parent of child	Step-child
c^{1-2}	Child of grandparent	Uncle or aunt
c^{-1+2}	Parent of grandchild	Son-in-law or Daughter-in-law
c^{-1+1-1}	Parent of child of parent	Step-parent
c^{-2+1}	Grandparent of child	Father-in-law or Mother-in-law
c^{-3}	Great-grandparent	

7. *Laws of the Index.* If two consecutive numbers in an index have the same sign, they can be added together; but, if they have different signs, they cannot be simplified by ordinary subtraction. Suppose the index is $p-q$, it may be irreducible, or it may reduce to $(p-1)-(q-1)$, or further to $(p-2)-(q-2)$, and so on until one of the indices is reduced to 0.

8. The general relationships are rendered more specific by introducing before a c or c^{-1} one of the sex-symbols m and f, the former denoting *male*,

the latter *female*. Thus mc denotes son, fc daughter, mc^{-1} father, fc^{-1} mother, $mcmc^{-1}$ son of the father, $mcfc^{-1}$ son of the mother.

9. A general relationship may be expanded into specific relationships by putting in $m+f$ before one or more of the symbols.

Thus
$$\begin{aligned}\Sigma c^2\mathrm{A} &= \Sigma c\,(m+f)\,c\mathrm{A} = \Sigma\,(cmc+cfc)\,\mathrm{A}\\ &= \Sigma\,(m+f)\,c^2\mathrm{A} = \Sigma\,(mc^2+fc^2)\,\mathrm{A}\\ &= \Sigma\,(m+f)\,c\,(m+f)\,c\mathrm{A} = \Sigma\,(mcmc+mcfc+fcmc+fcfc)\,\mathrm{A}.\end{aligned}$$

10. *Laws of Reduction.*—If the sex-symbols preceding and succeeding the expression c^{1-1} are the same, the expression may reduce to 1; but if the sex-symbols are different, the expression cannot reduce to 1. If the sex-symbols preceding and succeeding the expression c^{-1+1} are the same, the expression must reduce to 1; and if these are different, the expression cannot reduce to 1.

11. Name-symbols may be inserted before a relationship-symbol, as $c\mathrm{A}c^{-1}\mathrm{B}$, denoting a child of A who (that is, A) is a parent of B. When m or f is also specified, the proper position for the name-symbol is before the sex-symbol, as $c\mathrm{A}mc^{-1}\mathrm{B}f$, a child of A who is the father of B a woman.

12. A compound relationship due to the coexistence of several elementary relationships may be denoted by writing them after one another and using a dot to separate, as $cmc^{-1}\mathrm{A}\,.\,cfc^{-1}\mathrm{B}$, a child of the father of A, and also a child of the mother of B. If the terminal names be the same for all the elementary relationships, only the last need be written.

13. The Σ of a compound relationship is conditioned by the Σ's of the components. We may write $\Sigma\,(cm\mathrm{A}\,.\,cf\mathrm{B}) = \Sigma cm\mathrm{A}\,.\,\Sigma cf\mathrm{B}$. Here the Σ of the resultant term cannot be greater than the Σ of either component; it cannot be less than their sum minus the Σ of $\Sigma\mathrm{U}$; and it cannot be less than 0.

14. A compound relationship whose elementary relationships are specific forms of the same general relationship may be denoted by the expression for the general relationship with a vinculum over it and an index expressing the number of times the general relationship occurs. For example, $c^{\overline{1-1}^2}\mathrm{A} = cmc^{-1}\,.\,cfc^{-1}\mathrm{A}$. Thus $c^{\overline{1-1}^2}\mathrm{A}$ denotes a full brother or sister of A. Similarly $c^{\overline{1-1}^1}\mathrm{A}$ denotes a half brother or sister of A; and this notation gives us $c^{\overline{1-1}^0}\mathrm{A}$ as the expression for a non-brother or non-sister of A. This index does not coincide with the Boolian index; for the latter denotes the combination of the same relationship with itself.

15. Let r denote any general relationship, and suppose that it can assume n principal specific forms; then

$$\Sigma r\mathrm{A} = \Sigma\bar{r}^1 + 2\Sigma\bar{r}^2 + 3\Sigma\bar{r}^3 + \ldots + n\Sigma\bar{r}^n.$$

Several of these terms may be non-existent by reason of the Laws of Marriages.

16. *Transformation of a Conditional Equation.* — By a conditional equation is meant one which is true of a particular man as A or B, but not of any man U. For example, mc^{-1} Dick $= mc$ Tom. RULE.—*Any symbol may be taken from the front of one side of an equation, provided its reciprocal be placed in front of the other side.* The reciprocal of m is m, that of f is f. Also $mm = m$, $ff = f$, $mf = 0$. For example, in the case of the above equation, Dick $= cmc$ Tom; Tom $= c^{-1}mc^{-1}$ Dick. The reciprocal of Σ denoted by $\frac{1}{\Sigma}$ means *each of all*; $\frac{1}{3}$ means *each of three.*

17. *Transformation of a Universal Equation.*—By a Universal Equation

is meant one which is true for any person whatever, or for any person living under the laws of a given community. An example of the former is $\Sigma c \,.\, c^{-1}U = 0$; of the latter $\Sigma cm \,.\, cfcmU = 0$.

RULE.—*A universal equation may be transformed in accordance with the previous Rule, not only at the front of the factors, but also at their end.*

6653. (By W. R. WESTROPP ROBERTS, M.A.) — A heavy elliptical lamina is placed in contact with a vertical wall OA, and with a horizontal wall OB, and moves in a vertical plane. Show that, the walls being smooth, the lamina leaves the vertical wall when

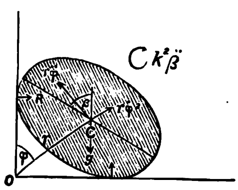
$$2(\sin\alpha - \sin\phi)\left\{1 + \frac{r^4a^2b^2\sin^2\phi\cos 2\phi}{(r^4\sin^2\phi\cos^2\phi - a^2b^2)(5r^4\sin^2\phi\cos^2\phi - 4a^2b^2)}\right\} = \sin\phi,$$

where $r^2 = a^2 + b^2$, $\phi = \widehat{COB}$, C being the centre of the lamina.

Solution by D. EDWARDES; CHARLOTTE A. SCOTT; *and others.*

Measuring ϕ so that $\frac{d\phi}{dt}$ is positive, and taking the mass of the body to be unity, the equation of energy is

$$\dot\phi^2 + \tfrac{1}{4}\left(\frac{d\beta}{dt}\right)^2 = \frac{2g}{r}(\cos\alpha - \cos\phi),$$



β being the inclination of major axis to the vertical at time t. But, by geometry, we have

$$\left(\frac{d\beta}{dt}\right)^2 = \frac{r^4\sin^2\phi\cos^2\phi}{(r^2\cos^2\phi - b^2)(a^2 - r^2\cos^2\phi)}\left(\frac{d\phi}{dt}\right)^2;$$

whence the equation above becomes

$$\dot\phi^2\left\{1 + \tfrac{1}{4}\,.\,\frac{r^4\sin^2\phi\cos^2\phi}{(r^2\cos^2\phi - b^2)(a^2 - r^2\cos^2\phi)}\right\} = \frac{2g}{r}(\cos\alpha - \cos\phi).$$

Reversing the effective forces, as in the figure, and resolving horizontally, we have $\qquad R + r\dot\phi^2\sin\phi = r\ddot\phi\cos\phi;$

therefore R vanishes when $\dot\phi^2\sin\phi = \ddot\phi\cos\phi$, that is, when

$$\dot\phi^2\left\{2\tan\phi\left[1 + \frac{r\sin^2\phi\cos^2\phi}{4(r^2\cos^2\phi - b^2)(a^2 - r^2\cos^2\phi)}\right] + \frac{r^4}{16}\frac{d}{d\phi}\left[\frac{\sin^2 2\phi}{(r^2\cos^2\phi - b^2)(a^2 - r^2\cos^2\phi)}\right]\right\} = \frac{2g}{r}\sin\phi,$$

or, performing the differentiation and substituting for $\dot\phi^2$ its value, when

$$2(\cos\alpha - \cos\phi)\left\{1 - \frac{r^4a^2b^2\cos^2\phi\cos 2\phi}{(r^4\sin^2\phi\cos\phi - a^2b^2)(5r^4\sin^2\phi\cos^2\phi - 4a^2b^2)}\right\} = \cos\phi,$$

which is equivalent to the PROPOSER'S result.

6013. (By Professor MINCHIN, M.A.)—**If at any point in a body subject to stress the principal stresses consist of two tensions of intensities A and B (A > B) and a pressure of intensity C, show that the maximum intensity of shearing stress is $\sqrt{AC}$, and find the plane on which it is exerted. If the principal stresses are a tension of intensity A and two pressures of intensities B and C (B > C), show that the maximum intensity of shearing stress is $\sqrt{AB}$, and find the plane on which it is exerted.**

Solution by W. H. BESANT, M.A., F.R.S.

The stress-quadric is $Ax^2+By^2-Cz^2=f$, and the plane $lx+my+nz=0$, with the condition $Al^2+Bm^2-Cn^2=0$, is a plane of shearing stress.

The tangent plane at (lr, mr, nr) is $Alrx+Bmry-Cnrz=f$. The perpendicular $p=f\div(A^2l^2+B^2m^2+C^2n^2)^{\frac{1}{2}}$, therefore the shearing stress $F=\dfrac{f}{pr}=(A^2l^2+B^2m^2+C^2n^2)^{\frac{1}{2}}$. We hence obtain, taking $l^2+m^2+n^2=1$, $F^2=AC-m^2(B+C)(A-B)$, which is a maximum when $m=0$; and then we have $l^2=\dfrac{C}{A+C}$, and $n^2=\dfrac{A}{A+C}$.

[Another solution is given in *Reprint*, Vol. XXXIII., pp. 80, 81.]

6700. (By Professor MOREL.)—On considère un cercle, un triangle inscrit ABC, et un triangle circonscrit A′B′C′, tel que les points de contact du second sont aux sommets du premier. D'un point quelconque M de la circonférence, on abaisse des perpendiculaires MP, MQ, MR sur les côtés du premier triangle et des perpendiculaires MP′, MQ′, MR′ sur les côtés du second. Démontrer que l'on a toujours

$$MP\,.\,MQ\,.\,MR = MP'.\,MQ'.\,MR'.$$

Solution by the Rev. T. R. TERRY, F.R.A.S.; G. HEPPEL, M.A.; *and others.*

Taking ABC as the triangle of reference, the equations to the circle and the side B′C′ are $a\beta\gamma+b\gamma+a\beta=0,\quad b\gamma+c\beta=0$ ....(1, 2); hence, if $(\alpha', \beta', \gamma')$ be the coordinates of M, a point on the circle, we have

$$\text{length } MP' = \pm\frac{b\gamma'+c\beta'}{(b^2+c^2-2bc\cos A)^{\frac{1}{2}}} = \frac{\beta'\gamma'}{\alpha'}, \text{ by (1);}$$

therefore $\qquad MP'.\,MQ'.\,MR' = \alpha'\beta'\gamma' = MP\,.\,MQ\,.\,MR.$

[If O be the centre of the circle, $\dfrac{MQ'}{MB}=\dfrac{MB}{2OB}$, $\dfrac{MP}{MB}=\dfrac{MC}{2OC}$,

therefore $\dfrac{MP}{MQ'}=\dfrac{MC}{MB}$; similarly $\dfrac{MQ}{MR'}=\dfrac{MA}{MC}$, $\dfrac{MR}{MP'}=\dfrac{MB}{MA}$;

therefore $\qquad MP\,.\,MQ\,.\,MR = MP'.\,MQ'.\,MR'.$]

6040. (By Professor LLOYD TANNER, M.A.)—If

$$b^{n/a} \equiv b\,(b-a)\,(b-2a)\,\ldots\,[b-(n-1)\,.\,a],$$

prove that
$$\begin{vmatrix} b^{1/b}, & 0, & 0, & \ldots, & b^{n/a} \\ (2b)^{1/b}, & (2b)^{2/b}, & 0, & \ldots, & (2b)^{n/a} \\ (3b)^{1/b}, & (3b)^{2/b}, & (3b)^{3/b}, & \ldots, & (3b)^{n/a} \\ \ldots & . & \ldots & \ldots & \ldots \\ (nb)^{1/b} & (nb)^{n/b}, & (nb)^{3/b}, & \ldots, & (nb)^{n/a}-(nb)^{n/b} \end{vmatrix} = 0.$$

Solution by the PROPOSER.

We may write

$$c^{n/a} = p_0 + p_1\,c^{1/b} + p_2\,c^{2/b} + \ldots + p_{n-1}\,c^{n-1/b} + p_n\,c^{n/b},$$

the coefficients p being independent of c. For each side is a polynomial of the n^{th} degree in c, and p_n, p_{n-1}, &c. can be determined in succession by comparing coefficients of c^n, c^{n-1}, &c. on each side. In particular $p_n = 1$; also $p_0 = 0$, since c is a factor of every other term. Thus

$$p_1\,c^{1/b} + p_2\,c^{2/b} + \ldots + p_{n-1}\,c^{n-1/b} - (c^{n/a} - c^{n/b}) = 0.$$

If we form n equations from this by giving c, n different values, we can eliminate p_1, p_2, &c. The relation given in the question is obtained by putting b, $2b$, $3b$, &c. in turn for c, noticing that $(rb)^{s/b} = 0$ when $s > r$.

6175. (By the Rev. W. ROBERTS, M.A.)—P, Q, R are three points on an equilateral hyperbola whose centre is O, such that OQ bisects the angle POR; M is the middle point of the chord PQ, and N the middle point of the chord QR; express the ratio $\dfrac{\text{OM}\,.\,\text{PQ}}{\text{ON}\,.\,\text{QR}}$ in terms of the ratio $\dfrac{\text{OP}}{\text{OR}}$.

Solution by W. J. C SHARP, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

Let $x^2 - y^2 = a^2$ be the equation to the hyperbola, and θ, $\frac{1}{2}(\theta+\phi)$, ϕ the angles made by OP, OQ, OR respectively with the axis; then we have

$$\text{OP}^2 = a^2 \sec 2\theta, \quad \text{OQ}^2 = a^2 \sec(\theta+\phi), \quad \text{OR}^2 = a^2 \sec 2\phi,$$

$$\text{PQ}^2 = \text{OP}^2 + \text{OQ}^2 - 2\,.\,\text{OP}\,.\,\text{OQ} \cos \tfrac{1}{2}(\theta-\phi),$$

$$4\,.\,\text{OM}^2 = \text{OP}^2 + \text{OQ}^2 + 2\,.\,\text{OP}\,.\,\text{OQ} \cos \tfrac{1}{2}(\theta-\phi);$$

therefore $4\,.\,\text{OM}^2\,.\,\text{PQ}^2 = \text{OP}^4 + \text{OQ}^4 - 2\,.\,\text{OP}^2\,.\,\text{OQ}^2 \cos(\theta-\phi)$;

similarly $4\,.\,\text{ON}^2\,.\,\text{QR}^2 = \text{OQ}^4 + \text{OR}^4 - 2\,.\,\text{OQ}^2\,.\,\text{OR}^2 \cos(\theta-\phi)$;

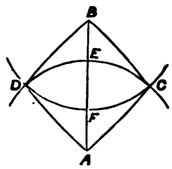
$$\therefore \left(\frac{\text{OM}\,.\,\text{PQ}}{\text{ON}\,.\,\text{QR}}\right)^2 = \frac{\sec^2 2\theta + \sec^2(\theta+\phi) - 2\sec 2\theta \sec(\theta+\phi)\cos(\theta-\phi)}{\sec^2 2\phi + \sec^2(\theta+\phi) - 2\sec 2\phi \sec(\theta+\phi)\cos(\theta-\phi)}$$

$$= \frac{\sec^2 2\theta}{\sec^2 2\phi} \cdot \frac{\cos^2(\theta+\phi) + \cos^2 2\theta - 2\cos 2\theta \cos(\theta+\phi)\cos(\theta-\phi)}{\cos^2(\theta+\phi) + \cos^2 2\phi - 2\cos 2\phi \cos(\theta+\phi)\cos(\theta-\phi)}$$

$$= \frac{\sec^2 2\theta}{\sec^2 2\phi} = \frac{\text{OP}^4}{\text{OR}^4}; \text{ therefore } \frac{\text{OM}\,.\,\text{PQ}}{\text{ON}\,.\,\text{QR}} = \frac{\text{OP}^2}{\text{OR}^2}.$$

6193. (By Professor SEITZ, M.A.)—Two equal small circles are drawn so as to intersect on the surface of a sphere of radius r; show that the average area of the spheric surface common to the two equal segments cut from the sphere is $(3\pi-8)\,r^2$.

Solution by the PROPOSER.



Let A and B be the poles of the two circles, C and D their intersections. Join AC, AD, BC, BD, and AB by arcs of great circles. Now, giving AC or BC, and AB all the values within the limits of the question, we are required to find the average area of the surface CEDF; and we may evidently consider one of the poles, as A, to be fixed. Let arc $AC = BC = rx$, $AB = ry$, $\angle CAB = \theta$, $\angle ACB = 2\phi$; then we have

$$\tan \tfrac{1}{2}y = \tan x \cos\theta,\quad \sin\tfrac{1}{2}y = \sin x \sin\phi,$$

$$\text{area CEDF} = \text{area ACED} + \text{area BCFD} - \text{area ACBD} = 2r^2(\pi - 2\phi - 2\theta\cos x);$$

hence the required average is

$$\int_0^{\frac{1}{2}\pi}\int_0^{2x} 2r^2(\pi-2\phi-2\theta\cos x)\, r\,dx\,.\,2\pi r^2 \sin y\,dy \div \int_0^{\frac{1}{2}\pi}\int_0^{2x} r\,dx\,.\,2\pi r^2\sin y\,dy$$

$$= \frac{4r^2}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{2x}(\pi-2\phi-2\theta\cos x)\,dx\sin y\,dy = 16r^2\int_0^{\frac{1}{2}\pi}\sin^4\tfrac{1}{2}x\,dx = r^2(3\pi-8).$$

6059. (By W. J. C. SHARP, M.A.)—If tangents be drawn to each of the cubics $\lambda U + \mu H$, from its points of intersection with the line $\alpha x + \beta y + \gamma z = 0$, prove that the points of contact all lie on the same quartic.

Solution by the PROPOSER.

Since the tangential of (x, y, z) is the intersection of

$$U_1\xi + U_2\eta + U_3\zeta = 0 \text{ and } H_1\xi + H_2\eta + H_3\zeta = 0$$

(SALMON's *Higher Plane Curves*, p. 152),

the quartic
$$\begin{vmatrix} U_1 & U_2 & U_3 \\ H_1 & H_2 & H_3 \\ \alpha & \beta & \gamma \end{vmatrix} = 0$$

must pass through the points of contact of the tangents which can be drawn to $U = 0$, from its points of intersection with $\alpha x + \beta y + \gamma z = 0$, and from the form of the equation it is identical with the corresponding quartic for any cubic of the form $aU + bH = 0$, including the Hessian itself.

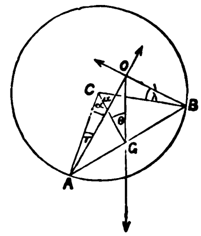
6776. (By W. H. H. HUDSON, M.A.)—A uniform rod, of length $2a \sin \alpha$, is placed within a rough vertical circle, of radius a, and is on the point of motion, the coefficient of friction at its upper and lower ends being $\tan \lambda'$, $\tan \lambda$; prove that, if θ be the inclination to the vertical of the line joining the centre of the sphere to the centre of the rod,

$$\tan \theta = \frac{\sin (\lambda + \lambda')}{2 \cos (\alpha + \lambda) \cos (\alpha - \lambda')};$$

and examine the case where $\alpha + \lambda = \frac{1}{2}\pi$.

Solution by W. B. GROVE, B.A.; D. EDWARDES; *and others.*

Let C be the centre of the circle, AB the rod, O the point of intersection of the total resistances at A and B. From the triangle AOB,



$$\frac{\sin \mathrm{OBG}}{\sin \mathrm{OAG}} = \frac{\cos (\theta - \mathrm{OBG})}{\cos (\theta + \mathrm{OAG})},$$

or $\qquad 2 \cot \mathrm{OGB} = \cot \mathrm{OAG} - \cot \mathrm{BOG}$,

or $\qquad 2 \tan \theta = \tan (\alpha + \lambda) + \tan (\alpha - \lambda')$,

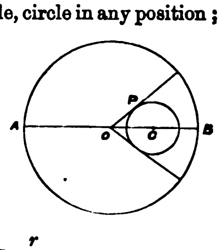
or $\qquad 2 \tan \theta = \dfrac{\sin (\lambda + \lambda')}{\cos (\alpha + \lambda) \cos (\alpha - \lambda')}$.

If $\lambda = \frac{1}{2}\pi - \alpha$, the angle of friction at A is $\angle$ CAB. Hence the rod will rest in any position between the limits $\theta = 0$ and $\theta = \frac{1}{2}\pi$ inclusive.

6751. (By the EDITOR.)—A circle of given radius is drawn at random in a given circle; find the chance that a radius drawn at random in the fixed circle will cut the other.

Solution by Professor MATZ, M.A.; BELLE EASTON; *and others.*

Let O be the centre of fixed, C the centre of variable, circle in any position; AOB the diameter through C; R and r the radii of these circles; OP a tangent; $\angle \mathrm{POB} = \theta$; $\mathrm{OC} = x$. Then, while C is fixed, the chance is $\dfrac{\theta}{\pi}$; hence, since $p = 1$, when $\mathrm{OC} < r$, we have



$$p = \int_r^{\mathrm{R}-r} \frac{\theta}{\pi} \cdot \frac{2\pi x\, dx}{\pi (\mathrm{R}-r)^2} + \frac{r^2}{(\mathrm{R}-r)^2}$$

$$= \frac{1}{\pi (\mathrm{R}-r)^2} \left\{ \theta x^2 - \int x^2\, d\theta \right\}_r^{\mathrm{R}-r} + \frac{r^2}{(\mathrm{R}-r)^2}$$

$$= \frac{1}{\pi} (\alpha + \cos \alpha \sin \alpha) + \tfrac{1}{2} \sin^2 \alpha, \text{ where } \sin \alpha = \frac{r}{\mathrm{R}-r}.$$

If $r = \frac{1}{3}\mathrm{R}$, $\alpha = \frac{1}{2}\pi$, and $p = 1$, as it should.

6507. (By Professor WOLSTENHOLME, M.A.)—If $a_1, a_2 \ldots, b_1, b_2 \ldots$ be all positive quantities, prove that

$$\int_0^\infty \frac{\sin a_1 x \sin a_2 x \ldots \sin a_n x \cos b_1 x \cos b_2 x \ldots \cos b_m x \sin \lambda x}{x^{n+1}}\, dx$$

will be $\frac{1}{2}\pi a_1 a_2 \ldots a_n$, if λ have any value not less than $\Sigma(a) + \Sigma(b)$.

[The value of the integral for all positive values of λ is positive, and increases as λ increases from 0 to the critical value above given. Thus the values of $\int_0^\infty \frac{\sin x \sin 2x \cos 3x \cos 4x \sin \lambda x}{x^3}\, dx$, when λ has the values 0, 1, 2 ... 10, are as 0, 7, 12, 15, 16, 17, 20, 24, 28, 31, 32; its value for all values of λ greater than 10 being π.]

Solution by the PROPOSER; C. B. S. CAVALLIN, M.A.; *and others.*

If $u = \sin a_1 x \sin a_2 x \ldots \sin a_n x \sin \lambda x$, it is obvious that the lowest power of x involved in u, when expressed as a series in ascending powers of x, is x^{n+1}; hence the quantities $\frac{u}{x^n}, \frac{\frac{du}{dx}}{x^{n-1}}, \frac{\frac{d^2u}{dx^2}}{x^{n-2}}$, &c. ... will all vanish when $x = 0$, and they will also vanish when $x = \infty$. Hence, by integrating by parts n times, we see that $\int_0^\infty \frac{u}{x^{n+1}}\, dx = \frac{1}{n!}\int_0^\infty \frac{\frac{d^n u}{dx^n}}{x}\, dx.$

Now $2 \sin \lambda x \sin a_1 x = \cos(\lambda - a_1) x - \cos(\lambda + a_1) x_1$,

$$4 \sin \lambda x \sin a_1 x \sin a_2 x = \sin(\lambda - a_1 + a_2) x \sin(\lambda - a_1 - a_2) x - \sin(\lambda + a_1 + a_2) x + \sin(\lambda + a_1 - a_2) x,$$

$$8 \sin \lambda x \sin a_1 x \sin a_2 x \sin a_3 x = \cos(\lambda - a_1 + a_2 - a_3) x - \cos(\lambda - a_1 + a_2 + a_3) x + \ldots,$$

and so on. In general, when n is even, $2^n \sin \lambda x \sin a_1 x \sin a_2 x \ldots \sin a_n x$ will be the sum of a number of terms each of which is of the form $\pm \sin(\lambda \pm a_1 \pm a_2 \ldots \pm a_n) x$, the sign of the term being regulated by the number of negative terms in the argument,—being positive when that number is odd and negative when that number is even if $\frac{1}{2}n$ be odd, but positive when that number is even and negative when that number is odd if $\frac{1}{2}n$ be even; when n is odd, $2^n u$ will be the sum of a number of terms of the form $\pm \cos(\lambda \pm a_1 \pm a_2 \ldots \pm a_n) x$, the sign of term being regulated as before with $\frac{1}{2}(n-1)$ in place of $\frac{1}{2}n$. Hence $\frac{d^n u}{dx^n}$ will in all cases be the sum of a number of terms of the form $\pm k^n \sin kx$, where k is one of the values of $(\lambda \pm a_1 \pm a_2 \ldots \pm a_n)$, and the sign of the term is + if the number of negative terms in the argument is even or zero, and − if this number is odd. Hence the integral is $\frac{1}{2}\pi \frac{1}{2^n n!} \times$ the sum of a number of terms each of which is of the form $\pm(\lambda \pm a_1 \pm a_2 \ldots \pm a_n)^n$, the rule for the pre-

fixed sign being the same as before when the coefficient $\lambda \pm a_1 \pm a_2 \ldots \pm a_n$ is positive, but the prefixed sign must be reversed when the coefficient is negative, since $\int_0^\infty \frac{\sin kx}{x} dx$ is $\pm \frac{1}{2}\pi$ according as k is positive or negative.

Hence, for all values of λ greater than $\Sigma(a)$, all these coefficients will be positive, and in that case the sum is $2^n n!\, a_1 a_2 \ldots a_n$, and the value of the integral is $\frac{1}{2}\pi\, a_1 a_2 \ldots a_n$. As λ decreases from $\Sigma(a)$ to 0, the form of the integral will change at each transit of λ through a value $-k$ which makes one of the coefficients vanish, each new form being deduced from the preceding by subtracting from the previous one the term $(-1)^p 2(\lambda+k)^n$, where p is the number of negative signs in k. I am not at present in a condition to prove that the integral always increases with λ from $\lambda = 0$ to $\lambda = \Sigma(a)$; but have found it to be so in all the cases tried.

Next, $2u \cos b_1 x = \sin a_1 x \ldots \sin a_n x [\sin(\lambda + b_1)x + \sin(\lambda - b_1)x] \equiv u_1 + u_2$;

and, by the former part, $\int_0^\infty \frac{u_1}{x^{n+1}} dx = \int_0^\infty \frac{u_2 dx}{x^{n+1}} = \int_0^\infty \frac{u dx}{x^{n+1}}$, if $\lambda - b_1 > \Sigma(a)$, or if $\lambda > b_1 + \Sigma(a)$. Hence, under this condition,

$$\int_0^\infty \frac{\sin a_1 x \ldots \sin a_n x \cos b_1 x \sin \lambda x}{x^{n+1}} dx = \tfrac{1}{2}\pi\, a_1 a_2 \ldots a_n.$$

In the same manner it is obvious that the introduction of any number of factors $\cos b_2 x_1, \ldots \cos b_m x$ will not affect the value of the integral, provided λ has a value greater than $\Sigma(b) + \Sigma(a)$.

It has been tacitly assumed throughout that the symbols a, b always represent positive quantities.

[The theorem, assumed above for shortness' sake, that

$$\Sigma \pm (\lambda \pm a_1 \pm a_2 \ldots \pm a_n)^n$$

is equal to $2^n n!\, a_1 a_2 \ldots a_n$, is very readily proved. Since any one of the a symbols (a_1) occurs just as often with the positive as with the negative sign, and since the sign of the term depends upon the number of negative constituents, any two terms which differ only in the sign of a_1 will have opposite signs, and when $a_1 = 0$ will destroy each other. Hence the whole is divisible by each of the n factors $a_1, a_2 \ldots a_n$, and being of n dimensions can have no other factors. Also in each term the coefficient of $a_1 a_2 \ldots a_n$ is $+n!$ since the sign prefixed is $(-1)^p$ and each product $a_1 a_2 \ldots a_n$ is again multiplied by $(-1)^p$, p being the number of negative constituents. The number of terms is 2^n; hence the coefficient of $a_1 a_2 \ldots a_n$ is $2^n n!$; and the whole sum is $n!\, 2^n a_1 a_2 \ldots a_n$.

The sum of the 2^{m+n} terms, each of which is of the form

$$(-1)^p (x \pm a_1 \pm a_2 \ldots \pm a_n \pm b_1 \pm b_2 \ldots \pm b_m)^n,$$

where p is the number of negative a constituents, is $n!\, 2^{m+n} a_1 a_2 \ldots a_n$.]

5941. (By L. H. Rosenthal, M.A.)—Find, for the biquadratic whose roots are $\alpha, \beta, \gamma, \delta$, the equation whose roots are $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$, &c.

Solution by J. L. McKenzie, B.A.; Sarah Marks; *and others.*

Let the given biquadratic be $x^4+px^3+qx^2+rx+s=0$;

then $\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma = \Sigma(x^3)-\delta^3-\dfrac{3s}{\delta} = -p^3+3pq-3r-\delta^3-\dfrac{3s}{\delta}$;

hence let $$y=-p^3+3pq-3r-x^3-\frac{3s}{x},$$

and eliminate x between this biquadratic and the given one. Or thus:—

$$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha),$$

and $$\alpha+\beta+\gamma=-p-\delta,\quad \alpha^2+\beta^2+\gamma^2=p^2-2q-\delta^2,$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=q-\delta(\alpha+\beta+\gamma)=q+p\delta+\delta^2;$$

therefore put $$y=(-p-x)(p^2-2q-x^2-q-px-x^2),$$

or $$2x^3+3px^2+3qx+(3pq-p^3-y)=0;$$

and eliminate x between this and the given biquadratic. The work will be rather long; the resultant of a biquadratic and a cubic contains 76 terms.

5773 & 5950. (By J. L. McKenzie, B.A.)—Through any point P on a circular cubic, draw any circle, cutting the cubic again in A, B, C; through A, B draw any circle cutting the cubic in D, E; let PD cut the cubic in Q, QC in R, and RE in S; prove that PS is the tangent to the cubic at P.

Prove the following constructions to find the chord of curvature at a given point P on a circular cubic:—(1) Take any point A on the cubic, and draw a line through A parallel to the real asymptote, cutting the cubic in B. Let the tangent at P cut the cubic in C; draw AP cutting the cubic in D, CD in E, and BE in Q. Then Q is the point in which the circle of curvature at P cuts the cubic again. (2) Let the real asymptote cut the cubic in F; a line through P parallel to the asymptote, in G, and the tangent at P in C. Draw CG cutting the cubic in H, and HF in Q. Then, as before, PQ is the chord of curvature at P.

Solution by the Proposer.

The theorems in both of these questions may be proved by the method of "residuation."

5773. Consider the system of 7 points A, B, D, P, P, I, J (I and J being the circular points). The circle ABP and the line PD form one cubic through these points, and cutting the given cubic again in C and Q; the line QC must therefore pass through R, the "point-residual" of the 7 points. Again, the circle ABD and the line PP (the tangent at P), form another cubic through the 7 points, cutting the given cubic again in E and S; therefore ES passes through R. This proves the theorem.

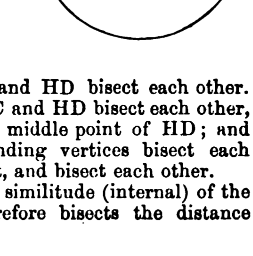
5950. (1) Let X be the point at infinity at which the asymptote touches the cubic. Take the system of 7 points on the cubic P, P, P, I, J, X, and

A. The circle of curvature at P and the line AX parallel to the asymptote form a cubic through these 7 points, cutting the given cubic again in B, and in the required point Q. Therefore BQ passes through E, the "point-residual" of the 7 points. Again, the tangent PP, the line at infinity XIJ, and the line CD form another cubic through the 7 points, cutting the given cubic again in C and D; therefore CD must pass through E. Hence the construction follows.

(2) Take now, as the 7 points, P, P, P, X, X, I, J; then the circle of curvature at P and the line XX (the asymptote) form a cubic through the 7 points, cutting the given cubic again in F and Q. And the tangent PP, the line PX parallel to the asymptote, and the line at infinity XIJ, form another cubic through the 7 points, cutting the given cubic again in C and G. Hence the lines FQ and CG must both meet the given cubic in the same point H,—the "point-residual" of the 7 points.

6403. (By D. Edwardes.)—If ABCD be a quadrilateral inscribed in a circle, and H, K, L, M are the orthocentres of the triangles formed by its sides and diagonals; prove that, (1) HKLM is a quadrilateral equal and similar to ABCD; (2) A, B, C, D are the orthocentres of the triangles formed from the sides and diagonals of HKLM; (3) the lines joining similar angular points meet in a point and mutually bisect one another; and (4) that this point bisects also the line joining the centres of the circles ABCD and HKLM.

Solution by C. Morgan, B.A.; G. F. Walker, M.A.; *and others.*



1. The distances AH and DK are each double of the distance of O from BC (O being the centre of the circle ABCD); hence, AH and DK being equal and parallel, HD, HK are equal and parallel. Similarly for the other sides of the quadrilateral. Hence the quadrilateral HKLM is equal in all respects to ABCD.

2. Since HM is parallel to CD, AL, which is perpendicular to DC, is also perpendicular to MH. Similarly, AH is perpendicular to ML, therefore A is orthocentre of HLM; and so for the other triangles formed by the sides and diagonals of HKLM.

3. Since AHKD is a parallelogram, AK and HD bisect each other. Similarly, since CHMD is a parallelogram, MC and HD bisect each other, therefore AK and MC bisect each other at the middle point of HD; and similarly any pair of lines joining corresponding vertices bisect each other, and must therefore all meet in one point, and bisect each other.

4. This point is evidently, then, a centre of similitude (internal) of the two equal circles ABCD, HKLM, and therefore bisects the distance between their centres.

6627. (By Professor TOWNSEND, F.R.S.)—Two shallow circular arcs, of arbitrary versed-sines, being supposed described on either segment, and on the same produced outwards to half its length, of a uniform elastic rod, supported in a horizontal position by one central and two terminal props, and bent slightly by its own weight; show that the vertical depression varies as the product of the ordinates to the arcs throughout the entire extent of the segment.

Solution by the PROPOSER.

Taking the horizontal and vertical lines through the centre of the rod, in the vertical plane of its equilibrium, as axes of x and of y respectively, and, in the integral of the familiar differential equation of the curve of strain of its axis under the action of gravity, viz., $\frac{d^4y}{dx^4}=k$, (where k is a constant depending on its density, elasticity, and transverse section,) determining the values of the four arbitary constants from the conditions that $y=0$ and $\frac{dy}{dx}=0$ for $x=0$, and that $y=0$ and $\frac{d^2y}{dx^2}=0$ for $x=a$, where a is its semi-length; we get without difficulty, for the equation in finite terms of its strained axis, $y=kx^2(a-x)(\frac{3}{2}a-x)$; from which the property immediately follows, the ordinates to the two arcs, whatever be their versed-sines, varying as the two products $x(a-x)$ and $x(\frac{3}{2}a-x)$ in the two cases respectively; and therefore, &c.

5939. (By R. A. ROBERTS, M.A.) — In the motion of a material particle constrained to move on a sphere, and attracted according to the law of the inverse fifth power of the distance by a uniform circular plate whose circumference is on the sphere, if the velocity in any position be that from infinity under the action of the force, prove that the orbit will be a circle orthogonal to the circumference of the plate.

Solution by the PROPOSER.

For the potential V of the plate at any point of space, we have $V=\frac{1}{4}\frac{m}{(\mu^2-c^2)(\mu^2-\nu^2)}$, where μ is half the sum and ν half the difference of the greatest and least distances from the circumference of the plate. Hence, at any point of a sphere passing through the circumference of the plate, V varies inversely as the square of the distance from the plane of the plate. But, if v be the velocity of the particle, v^2 varies as V, and $\int v\,ds$ is a minimum; or, using polar coordinates on the sphere, we have

$$\int\frac{(d\theta^2\cdot\sin^2\theta\,d\phi^2)^{\frac{1}{2}}}{\cos\alpha-\cos\theta}\text{ a maximum, giving }\frac{\sin^2\theta}{\left\{\frac{d\theta^2}{d\phi^2}+\sin^2\theta\right\}^{\frac{1}{2}}}=C(\cos\alpha-\cos\theta),$$

which represents a circle cutting the circle $\theta=\alpha$ orthogonally.

This theorem is also true, if we substitute for the plate the spherical surface bounded by a small circle.

[Mr. ROBERTS remarks that Mr. SHARP's solution of Quest. 5984 (*Reprint*, XXXII., p. 42) is altogether incorrect; since a value of V has been therein obtained different from what is given above, and that the process, furthermore, makes the resultant pass through the pole of the plane of the plate with regard to the sphere, which is not the case.]

6185. (By E. W. SYMONS, B.A.)—The vertex of a parabola moves along the pedal of a given curve, while its focus is fixed at the pole; prove that its envelop is the first negative pedal of the given curve.

Solution by H. HAYASH, M.A.; E. RUTTER; *and others.*

We begin by stating the known theorem, that the envelop of circles described on the radii vectores of a given curve is the pedal of that curve, thus,—the envelop of a circle passing through the pole, and whose centre lies on the radius vector of a given curve at a distance from the pole equal to half the length of that radius vector, is the pedal of the given curve, *i.e.*, the locus of foot of perpendicular let fall from the pole on any tangent to the given curve.

Reciprocation of this with respect to the pole gives the following theorem: The envelop of a parabola with focus at the pole, directrix parallel to any tangent to a given curve, at double the distance from the pole that the said tangent is, will be the envelop of a line drawn through the point of contact of the tangent and perpendicular to the line joining that point of contact to the pole, *i.e.*, will be the first negative pedal. Stated in different words, this is the theorem enunciated.

6640. (By R. RAWSON.)—Assuming that

$$u = \{\phi_1(x)\,\theta(x,y)+1\}^2 - \phi_2(x) = (x-y_1)(x-y_2)\,\ldots\,(x-y_n) \quad \ldots\ldots (1),$$

where u, $\phi_1(x)$, $\theta(x,y)$, and $\phi_2(x)$ are rational in x; show that

$$\int \frac{\psi(y_1)\,dy_1}{\phi_1(y_1)\,[\phi_2(y_1)]^{\frac{1}{2}}} + \ldots\ldots \int \frac{\psi(y_n)\,dy_n}{\phi_1(y_n)\,[\phi_2(y_n)]^{\frac{1}{2}}} = \frac{1}{x}\,\Big|\,\frac{\epsilon\psi(x)}{\phi_1(x)\,[\phi_2(x)]^{\frac{1}{2}}}$$
$$\times \log \frac{\phi_1(x)\,\theta(x,y)+1-[\phi_2(x)]^{\frac{1}{2}}}{\phi_1(x)\,\theta(x,y)+1+[\phi_2(x)]^{\frac{1}{2}}} \quad \ldots\ldots\ldots (2),$$

where $\epsilon = \mp 1$, $x \,|\, f(x)$ is used as a convenient symbol to represent the coefficient of x in the development of $f(x)$, and $x^{-1} \,|\, f(x)$ is the coefficient of x^{-1} in the development of $f(x)$, &c. [The $\psi(x)$ may be either rational or irrational; when $\psi(x)$ is rational, and $\phi_1 x = x - a$, the above integral becomes that determined by ABEL's theorem.]

Solution by the PROPOSER.

Differentiating (1), we have

$$dx = -2\left\{\phi_1(x)\,\theta(x,y)+1\right\}\phi_1(x)\frac{d}{dy}\,\theta(x,y)\,dy+\frac{du}{dx}$$

$$= \pm\, 2\phi_1(x)\left\{\phi_2(x)\right\}^{\frac12}\frac{d}{dx_1}\,\theta(x,y)+\frac{du}{dx},$$

$$\therefore\qquad \frac{\psi(x)\,dx}{\phi_1(x)\left\{\phi_2(x)\right\}^{\frac12}} = \frac{2\epsilon\,\psi(x)\dfrac{d}{dy}\,\theta(x,y)}{\dfrac{d}{dx}F(x,y)},\ \text{when } u = F(x,y)\text{.........}\ (3).$$

Now (3) is true for every value of x which satisfies (1). Hence, by integrating (3), and putting for x the values $y_1, y_2, \ldots y_n$ successively, there results

$$\int\frac{\psi(y_1)\,dy_1}{\phi_1(y_1)\left\{\phi_2(y_1)\right\}^{\frac12}}+\ldots\int\frac{\psi(y_n)\,dy_n}{\phi_1(y_n)\left\{\phi_2(y_n)\right\}^{\frac12}}$$

$$= 2\epsilon\int\left\{\frac{\psi(y_1)\dfrac{d}{dy}\,\theta(y_1,y)}{\dfrac{d}{dy_1}F(y_1,y)}+\ldots\frac{\psi(y_n)\dfrac{d}{dy}\,\theta(y_n,y)}{\dfrac{d}{dy_n}F(y_n,y)}\right\}dy+\text{C}\text{......}(4).$$

The right-hand side of (4) is a particular case of the property enunciated in Quest. 6559, where $m = 0$, $f(x) = 1$, and $\psi(x, y) = \psi(x)$. Therefore equation (4) becomes

$$\int\frac{\psi(y_1)\,dy_1}{\phi_1(y_1)\left\{\phi_2(y_1)\right\}^{\frac12}}+\ldots\int\frac{\psi(y_n)\,dy_n}{\phi_1(y_n)\left\{\phi_2(y_n)\right\}^{\frac12}}$$

$$= \frac{1}{x}\Bigg|\ 2\epsilon\int\frac{\psi(x)\dfrac{d}{dy}\,\theta(x,y)}{F(x,y)}\,dy+\text{C} = \frac{1}{x}\Bigg|\ 2\epsilon\,\psi(x)\int\frac{d\theta(x,y)}{F(x,y)}+\text{C}$$

$$= \frac{1}{x}\Bigg|\ \epsilon\,\frac{\psi(x)}{\phi_1(x)\left\{\phi_2(x)\right\}^{\frac12}}\ \log\frac{\phi_1(x)\,\theta(x,y)+1-\left\{\phi_2(x)\right\}^{\frac12}}{\phi_1(x)\,\theta(x,y)+1+\left\{\phi_2(x)\right\}^{\frac12}}+\text{C},$$

which proves the relation in the Question. The above demonstration assumes that the roots of (1) are unequal. ABEL has attempted to prove his theorems when some of the roots are equal, but his results appear to me to require reconsideration.

6500. (By G. F. WALKER, M.A.) — A chord PQ of the parabola $y^2 = 4ax$ passes through a fixed point $(c, 0)$ on the axes. Circles are described touching the parabola at P and Q, and passing through the focus; show that their second common point must lie on the curve

$$(x^2+y^2-ax)^2-3c\,(x-a)\,(x^2+y^2-ax)-\frac{c^2}{a}(3a-c)\,y^2 = 0.$$

Solution by I. H. TURRELL, M.A.; *the* PROPOSER; *and others.*

The equation to a circle touching at the point (x, y), and passing through the focus, is easily found to be

$$(1+m_1^2)(y^2-4ax) = \left(y-m_1x-\frac{a}{m_1}\right)(y+m_1x+3am_1),$$

or $$m_1^2(x^2+y^2)-ax(3+m_1^2)-ay\left(3m_1-\frac{1}{1}\right)+3a^2 = 0.$$

The circle touching at m_2, in same manner, is

$$m_2^2(x^2+y^2)-ax(3+m_2^2)-ay\left(3m_2-\frac{1}{m_2}\right)+3a^2 = 0;$$

and we want to eliminate m_1, m_2 from these and the condition $m_1m_2 = -\frac{a}{c}$.

Subtracting and dividing by m_1-m_2, we get

$$(m_1+m_2)(x^2+y^2-ax)-ay\left(3+\frac{1}{m_1m_2}\right) = 0 \quad\text{..................(1)}.$$

Multiply first by m_2^2, and subtract from the second multiplicand by m_1^2, and we get, after division by m_1-m_2,

$$-3ax(m_1+m_2)-3aym_1m_2+\frac{ay(m_1^2+m_1m_2+m_2^2)}{m_1m_2}+3a^2(m_1+m_2) = 0,$$

or $$\frac{ay(m_1+m_2)^2}{m_1m_2}-3a(x-a)(m_1+m_2)-ay(3m_1m_2+1) = 0.$$

Substituting from (1) for the last term, and dividing out by m_1+m_2, we get $$\frac{ay(m_1+m_2)}{m_1m_2}-3a(x-a)-m_1m_2(x^2+y^2-ax) = 0 \quad\text{...........(2)}.$$

Eliminating m_1+m_2 between (1) and (2), we get

$$\frac{a^2y^2}{m_1m_2}\left(3+\frac{1}{m_1m_2}\right) = (x^2+y^2-ax)\left[m_1m_2(x^2+y^2-ax)+3a(x-a)\right];$$

or, remembering $m_1m_2 = -\frac{a}{c}$, we find

$$-acy^2\left(3-\frac{c}{a}\right) = (x^2+y^2-ax)\left[-\frac{a}{c}(x^2+y^2-ax)+3a(x-a)\right],$$

$$c^2y^2(3a-c) = a(x^2+y^2-ax)\left[x^2+y^2-ax-3c(x-a)\right].$$

[The locus of the centre of either circle is $27ay^2 = (2x-a)(5a-x)^2$].

6735. (By Prince CAMILLE DE POLIGNAC.)—An unclosed polygon is inscribed in a conic and circumscribed about another; M_1, M_2 are two consecutive fixed sides; a_1, a_2 any other pair of consecutive sides taken in the same order; a_1 meets M_2 in m_2; and a_2 meets M_1 in m_1. If the line m_1m_2 passes through a fixed point, prove that the conics have double contact.

Solution by the PROPOSER.

Let any conic, and two tangents thereto, be

$$xy-z^2=0\equiv S,\quad x+\mu^2y-2\mu z=0,\quad x+\mu'^2y-2\mu'z=0\ldots\ldots(1,2,3);$$

then, tangent μ meets the axis of y in the point $(y=0,\ x=2\mu z)$; and tangent μ' meets the axis of x in the point $(x=0,\ y=2\mu'^{-1}z)$; and the condition that the line joining these two points should pass through a fixed point (α,β,γ) is $\begin{vmatrix}\alpha & \beta & \gamma\\ 2\mu & 0 & 1\\ 0 & 2 & \mu'\end{vmatrix}=0$, or $\beta\mu\mu'-2\gamma\mu+\alpha=0$ .... (4).

Eliminating $\mu,\ \mu'$ between (2), (3), (4), we get, for the locus of the intersection of tangents,

$$4\gamma^2xy+(\beta x+\alpha y)(\beta x+\alpha y-4\gamma z)=0,\ \text{or}\ S'\equiv 4\gamma^2S+(\beta x+\alpha y-2\gamma z)^2=0,$$

a conic which passes through the point (x,y), and has double contact with S.

Now, in virtue of equation (4), tangents $\mu,\ \mu'$ may be considered as two consecutive sides of the polygon; the same is true of x and y, which correspond respectively to the two consecutive values $\mu=0,\ \mu'=\infty$.

The polar of (α,β,γ) is the chord of contact $\beta x+\alpha y-2\gamma z=0$; hence the tangents common to S and S′ meet in (α,β,γ).

The lines through (α,β,γ) answering to each pair of consecutive sides trace out on the axes a homographic system of points, and the tangents through (α,β,γ) meet the axes in the foci of the system. For these are given by the equation $\beta\mu^2-2\gamma\mu+\alpha=0$, which expresses the condition that the point (α,β,γ) should lie on the tangent $x+\mu^2y-2\mu z=0$.

The anharmonic ratio of four consecutive tangents, or of four consecutive legs of the anharmonic pencil through (α,β,γ), is a constant quantity $=\dfrac{\alpha\beta}{4\gamma^2}$.

When the foci coincide, $\alpha\beta=\gamma^2$; the point (α,β,γ) lies on the conic S, and the two conics have a contact of the third order.

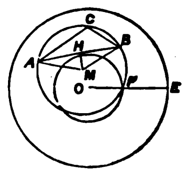
When the double contact is real, it becomes obvious, on drawing the figure, that the process of inscribing tangents converges towards the common tangents. In other words, the homographic points on the axes converge towards the foci.

This may be otherwise stated. The successive sides of the polygon correspond to the powers of a linear substitution, involved in the homographic equation (4). Hence the nth power of any linear substitution (to real foci) converges ultimately towards one focus, and the nth power of the inverse substitution towards the other.

1843. (By the EDITOR.)—(1) Three points being taken at random within a circle, prove that the chance that the circle circumscribing the triangle formed by joining them lies wholly within the given circle, is $\frac{2}{5}$; and (2) four points being taken at random within a sphere, prove that the chance that the sphere circumscribing the tetrahedron formed by joining them lies wholly within the given sphere, is $\dfrac{48\pi^2}{1925}$.

Solution by Professor SEITZ, M.A.

1. Let O be the centre of the given circle, ABC the triangle formed by joining the three random points A, B, C, and M the centre of the circle circumscribing ABC. From O as centre, with a radius OF equal to the difference between OE and AM, draw a circle.



Now, while AB and AC are given in length and direction, AM being less than OE, the area of the circle OF represents the number of ways the three points can be taken, so that the circle circumscribing the triangle will lie wholly within the given circle.

Let AB $= 2x$, OE $= r$, $\angle$AMH $= \theta$, $\angle$BAC $= \phi$, and $\psi =$ the angle which AB makes with a fixed line. Then we have

$$\text{AM} = x \operatorname{cosec} \theta, \quad \text{OF} = r - x \operatorname{cosec} \theta, \quad \text{AC} = 2x \sin(\theta - \phi) \operatorname{cosec} \theta;$$

an element of the circle at B is $4x\, dx\, d\psi$, and at C it is

$$d\,.\,(\text{AC}) \sin(\theta-\phi)\,.\,2x \operatorname{cosec}\theta\, d\phi = 4x^2 \operatorname{cosec}^3\theta \sin\phi \sin(\theta-\phi)\, d\theta\, d\phi.$$

The limits of θ are 0 and π, and doubled, since C may lie on either side of AB; those of ϕ are 0 and θ; of x, 0 and $r \sin\theta = x'$; and of ψ, 0 and 2π.

Hence, since the whole number of ways the three points can be taken is $\pi^3 r^6$, the chance in question is

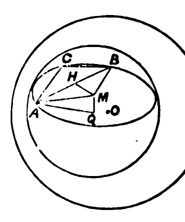
$$\frac{2}{\pi^3 r^6} \int_0^{\pi}\int_0^{\theta}\int_0^{x'}\int_0^{2\pi} \pi (r - x\operatorname{cosec}\theta)^2 \operatorname{cosec}^3\theta\, d\theta \sin\phi \sin(\theta-\phi)\, d\phi\,.16x^3\, dx\, d\psi$$

$$= \frac{64}{\pi r^6} \int_0^{\pi}\int_0^{\theta}\int_0^{x'} (r - x\operatorname{cosec}\theta)^2 \operatorname{cosec}^3\theta\, d\theta \sin\phi \sin(\theta-\phi)\, d\phi\, x^3\, dx$$

$$= \frac{16}{15\pi} \int_0^{\pi}\int_0^{\theta} \sin\theta\, d\theta \sin\phi \sin(\theta-\phi)\, d\phi$$

$$= \frac{8}{15\pi} \int_0^{\pi} (\sin\theta - \theta\cos\theta) \sin\theta\, d\theta = \frac{2}{5}.$$

2. Let O be the centre of the given sphere, A, B, C, D the four random points, Q the centre of the sphere whose surface contains A, B, C, D, and M the centre of the small circle of this sphere, which passes through A, B, C. Draw MH perpendicular to AB.



Now, while AB, AC, AD are given in length and direction, AQ being less than the radius of the given sphere, the volume of a sphere whose radius is equal to the difference of the given radius and AQ, represents the number of ways the four points can be taken, so that the sphere QA will lie wholly within the given sphere.

Let AB $= 2x$, $\angle$AMH $= \theta$, $\angle$BAC $= \phi$, $\angle$AQM $= \psi$, $\mu =$ the angle AB makes with a fixed line, and $\omega =$ the angle ABC makes with a plane through AB. Then we have

$$\text{AM} = x\operatorname{cosec}\theta, \quad \text{AQ} = x \operatorname{cosec}\theta \operatorname{cosec}\psi, \quad \text{AC} = 2x \sin(\theta-\phi)\operatorname{cosec}\theta;$$

an element of the sphere at B is $16\pi x^2 \sin\mu\, dx\, d\mu$, at C it is

$$d\,.\,(\mathrm{AC})\sin(\theta-\phi)\,.\,2x\,\mathrm{cosec}\,\theta\, d\phi\,.\,\mathrm{AC}\sin\phi\, d\omega$$
$$= 8x^3\,\mathrm{cosec}^4\theta\sin^2\phi\sin^2(\theta-\phi)\, d\theta\, d\phi\, d\omega,$$

and at D it is d (spher. seg. ABCD) $= \pi x^3\,\mathrm{cosec}^3\theta\,\mathrm{cosec}^4\psi\,(1-\cos\psi)^2\, d\psi$.

The limits of θ are 0 and π, and doubled, since C may lie on either side of AB; those of ϕ are 0 and θ; of ψ, 0 and π, and doubled, since D may lie on either side of the plane ABC; those of x are 0 and $r\sin\theta\sin\psi = x'$; of μ, 0 and π; and of ω, 0 and 2π. Hence, since the whole number of ways the four points can be taken is $(\frac{4}{3}\pi r^3)^4$, the chance in question is

$$\frac{4}{(\frac{4}{3}\pi r^3)^4}\int_0^\pi\int_0^\theta\int_0^\pi\int_0^{x'}\int_0^\pi\int_0^{2\pi} \tfrac{4}{3}\pi\,(r - x\,\mathrm{cosec}\,\theta\,\mathrm{cosec}\,\psi)^3\,\mathrm{cosec}^7\theta\, d\theta\sin^2\phi$$
$$\times\sin^2(\theta-\phi)\, d\phi\,.\,8\pi\,\mathrm{cosec}^4\psi\,(1-\cos\psi)^2\, d\psi\,.\,16\pi x^8\, dx\sin\mu\, d\mu\, d\omega$$

$$= \frac{864}{r^{14}}\int_0^\pi\int_0^\theta\int_0^\pi\int_0^{x'}(r - x\,\mathrm{cosec}\,\theta\,\mathrm{cosec}\,\psi)^3\,\mathrm{cosec}^7\theta\, d\theta\sin^2\phi\sin^2(\theta-\phi)\, d\phi$$
$$\times\,\mathrm{cosec}^4\psi\,(1-\cos\psi)^2\, d\psi\, x^8\, dx$$

$$= \frac{24}{55}\int_0^\pi\int_0^\theta\int_0^\pi \sin^2\theta\, d\theta\sin^2\phi\sin^2(\theta-\phi)\, d\phi\,(1-\cos\psi)^2\sin^5\psi\, d\psi$$

$$= \frac{1024}{1925}\int_0^\pi\int_0^\theta \sin^2\theta\, d\theta\sin^2\phi\sin^2(\theta-\phi)\, d\phi$$

$$= \frac{128}{1925}\int_0^\pi (3\theta - 2\theta\sin^2\theta - 3\sin\theta\cos\theta)\sin^2\theta\, d\theta = \frac{48\pi^2}{1925}.$$

[Other Solutions are given in *Reprints*, Vol. VIII., pp. 90—92; XIII., 17—21, 95—99; XVI., 50—53.]

6830. (By Professor CROFTON, F.R.S.)—Prove that

$$\epsilon^{\mathrm{D}+x^{-1}}\,\mathrm{F}(x) = \frac{x+1}{x}\,\mathrm{F}(x+1).$$

Solution by J. J. WALKER, M.A.

The formula of the Question may be generalized; viz., a, b being any constants, $\exp.\, a\left\{\mathrm{D} + (x+b)^{-1}\right\}\mathrm{F}(x) = \left\{1 + a(x+b)^{-1}\right\}\mathrm{F}(x+a)$.
To verify this, let v be a function of x satisfying $v^2 + a\mathrm{D}v = 0$; then, r being any integer, $(a\mathrm{D}+v)^r = a^r\mathrm{D}^r + ra^{r-1}v\mathrm{D}^{r-1}$. For, assuming this to hold for any value of r, it is easily shown that it holds for $r+1$. Hence, immediately, $f(a\mathrm{D}+r) = f(a\mathrm{D}) + vf'(a\mathrm{D})$; and in particular

$$\exp.\,(a\mathrm{D}+v)\,\mathrm{F}(x) = (1+v)\exp.\,(a\mathrm{D})\,\mathrm{F}(x) = (1+v)\,\mathrm{F}(x+a).$$

But $v^2 + a\mathrm{D}v = 0$ gives $\quad v = a(x+b)^{-1}$.

[Mr. THOMAS and others prove the theorem thus :—

$$(D+x^{-1})\,x^{n} = x^{-1}\,Dx^{n+1}, \text{ and } (D+x^{-1})^{m}\,x^{n} = x^{-1}\,D^{m}\,x^{n+1};$$

therefore $$f(D+x^{-1})\,\phi(x) = x^{-1} f(D)\,x\,\phi(x),$$

therefore $$e^{D+x^{-1}}\,F(x) = x \quad e^{D}\,x\,F(x) = \frac{x+1}{x}\,F(x+1).]$$

6795 & 6826. (By Professor SYLVESTER, F.R.S.)—(6795) Prove that $1+1\,.\,m+1\,.\,5\,\dfrac{m(m-1)}{1}+1\,.\,5\,.\,9\,\dfrac{m(m-1)(m-2)}{1\,.\,2\,.\,3}+\dots$ is divisible by 2^{m}. Ex.: $1+5+10\,(5)+10\,(5\,.\,9)+5\,(5\,.\,9\,.\,13)+5\,.\,9\,.\,13\,.\,17$

$$= 1+5+50+450+2925+9945 = 13376 = 2^{5}\times 418.$$

(6826.) Find the greatest common measure of D_n and D_{n+1}, where D_n represents a determinant of the n^{th} order of the form

$$\begin{vmatrix} 1 & 1 & . & . & . & . \\ 1 & 3 & 2 & . & . & . \\ . & 1 & 5 & 3 & . & . \\ . & . & 1 & 7 & 4 & . \\ . & . & . & 1 & 9 & 5 \\ . & . & . & . & 1 & 11 \end{vmatrix}$$

Solution by W. J. C. SHARP, M.A.; E. BUCK, B.A.; *and others.*

The determinants in 6826 satisfy the equation of differences

$$D_{n+1} = (2n+1\;\; D_n - n\,D_{n-1};$$

so that every measure of D_{n+1} and D_n measures D_{n-1}, and therefore the G. C. M. of D_{n+1} and D_n can differ from that of D_n and D_{n-1} by involving an additional factor which measures n. But every measure of n and D_{n+1} must measure the determinant obtained by writing

$$\begin{matrix} . & . & . & . & . & . & 1 & . & -1 & . & 0 \\ . & . & . & . & . & . & 0 & . & 1 & . & 1 \end{matrix}$$

instead of the last two lines of D_{n+1} and this $= -D_{n-1}$; therefore every measure of D_{n+1} and n measures D_{n-1}, and the G. C. M. of D_{n+1} and D_n is that of D_n and D_{n-1}, and ultimately of D_3 and D_2 or 2.

Again, denoting the series given in 6795 by u_m, we have

$$u_{m+1}-2u_m = -1\,.\,m-1\,.\,5\,\frac{m(m-3)}{1\,.\,2}-1\,.\,5\,.\,9\,\frac{m(m-1)(m-5)}{3\,!}-\&c.$$

$$= -m\left[1+1\,.\,5\left\{\frac{m-1}{2}-1\right\}+1\,.\,5\,.\,9\left\{\frac{(m-1)(m-2)}{3\,!}-\frac{m-1}{2\,!}\right\}\right.$$

$$\left.+1\,.\,5\,.\,9\,.\,13\left\{\frac{(m-1)(m-2)(m-3)}{4\,!}-\frac{(m-1)(m-2)}{3\,!}\right\}+\&c.\right]$$

$$= -m\left[-4-1.5.8\frac{m-1}{2!}-1.5.9.12\frac{(m-1)(m-2)}{3!}\ \&c.\right]$$

$$= 4m\left[1+1.5(m-1)+1.5.9\frac{(m-1)(m-2)}{2!}+\&c.\right]$$

$$= 4m\,(u_m-u_{m-1});$$

and therefore $u_{m+1} = (4m+2)\,u_m - 4mu_{m-1} = (2m+1)\,2u_m - 4mu_{m-1}$.
Hence it appears that if u_{m-1} is divisible by 2^{m-1} and u_m by 2^m, u_{m+1} is divisible by 2^{m+1}; and therefore, since $u_1=2$ and $u_2=8$, u_m is divisible by 2^m; also, if $u_m = 2^m q^m$, the equation of differences becomes $q_{m+1} = (2m+1)\,q_m - m\,q_{m-1}$, the equation satisfied by the determinants of Question 6826; and since $u_1 = 2$, $q_1 = 1 = D_1$, and $u_2 = 8$, $q_2 = 2 = D_2$, and the quantities q_m, &c. are the same as D_m, &c., which are therefore the denumerants mentioned in Prof. SYLVESTER's subjoined note.

From the process it appears that u_m is divisible by 2^{m+1}.

[Prof. SYLVESTER remarks that, if we call the quotient in Quest. 6795 q_m, $\{1.3.5\ldots(2m-1)\}\,q_m$ is the number of distinct terms in a general pure skew determinant of the order $2m$, so that q_m may be termed the *reduced* denumerant of the $2m^{\text{thic}}$ skew determinant. (See *American Journal of Mathematics*, No. 5.) Or, which is the same thing, q_m is the reduced denumerant of the number of distinct double duadic synthemes that may be formed with $2m$ elements. A double duadic syntheme of $2m$ elements, meaning a combination of $2m$ duads which is separable into 2 combinations of m duads, each of which latter contains all the $2m$ elements. Ex.: 1.2, 1.2, 3.4, 5.6, 3.5, 4.6, or 1.2, 3.4, 5.6, 2.3, 4.5, 1.6, is a double syntheme of 6 elements. So is 1.2, 3.4, 1.3, 2.4, 5.6, 7.8, 5.7, 6.8, or 1.2, 1.2, 3.4, 3.4, 5.6, 7.8, 5.7, 6.8, or 1.2, 2.3, 3.4, 4.5, 5.6, 6.7, 7.8, 1.8 of 8 elements.]

6779. (By the Rev. A. J. C. ALLEN, B.A.)—A uniform elastic rod is cut into three equal portions, of height h, and these are placed upright on a horizontal plane, the three being in the same vertical plane and at equal distances (l) from each other. A heavy uniform elastic rod, of length $2l$ and weight w, is placed on the top of them; prove that the pressures on the middle and either of the outside supports are

$$\frac{5}{8}\,w\left(1+\frac{24}{5}\cdot\frac{Kh}{El^3}\right)\div\left(1+9\cdot\frac{Kh}{El^3}\right),\quad \frac{3}{16}\,w\left(1+16\cdot\frac{Kh}{El^3}\right)\div\left(1+9\cdot\frac{Kh}{El^3}\right),$$

where K and E are the Young's modulus of the supports, and the flexural rigidity of the beam.

Solution by Prof. TOWNSEND, F.R.S.; G. F. WALKER, M.A.; *and others.*

These values, under a slightly different notation, have been given in the solution of my Question 4642 (see *Reprint*, Vol. XXIV., p. 25.) Thus, In the notation of the present question, that solution was as follows :—

Denoting by P the common value of the two terminal pressures, and by Q that of the central pressure, by p the common diminution in length of the two terminal props, and by q that of the central prop; then since, on elementary principles of longitudinal elasticity,

$$p : h = P : E, \quad q : h = Q : E,$$

and since evidently $2P + Q = W$, therefore

$$E(q-p) = (Q-P)\,h = \tfrac{1}{2}(3Q-W)\,h;$$

from which it appears that Q, and therefore P, would be known if $(q-p)$ were connected with it by another linear relation of the same form, which it may be shown it is as follows.

Since, on known principles of transverse deflection, $(q-p)$ cannot exceed positively $\frac{5}{48}\cdot\frac{l^3}{K}\cdot W$, which would be its value if the central prop were removed, in which case Q would $= 0$, and cannot exceed negatively $\frac{3}{48}\cdot\frac{l^3}{K}\cdot W$, which would be its value if the two terminal props were removed, in which case Q would $= W$; therefore, for every intermediate value of $(q-p)$, the corresponding value of Q, in accordance with the general principle that, under the same circumstances of production, all small effects are proportional to their causes, will be given by the proportion

$$Q : W = \left[\frac{5}{48}\cdot\frac{l^3}{K}\cdot W - (q-p)\right] : \frac{8}{48}\cdot\frac{l^3}{K}\cdot W;$$

which accordingly is the second linear relation connecting the two quantities in question.

Eliminating $(q-p)$ between the two linear equations connecting it with Q, we get at once for Q the value

$$\left[\frac{1}{3}\cdot\frac{l^3}{K}+3\frac{h}{E}\right]Q = \left[\frac{10}{48}\cdot\frac{l^3}{K}+\frac{h}{E}\right]W;$$

and, since $P = \frac{1}{2}(W-Q)$, consequently for P the value

$$\left[\frac{1}{3}\cdot\frac{l^3}{K}+3\frac{h}{E}\right]P = \left[\frac{3}{48}\cdot\frac{l^3}{K}+\frac{h}{E}\right]W;$$

which are manifestly identical with those of the question.

In the extreme case when $K = \infty$, that is, when the supported rod is perfectly rigid, then $P = Q = \frac{1}{3}W$; which are the known values of P and Q in that case.

In the other extreme case when $E = \infty$, that is, when the supporting props are perfectly rigid, then $P = \frac{3}{16}W$, and $Q = \frac{10}{16}W$; which are their known values in that case also.

6763. (By the Rev. C. TAYLOR, D.D.)—If the axes of one rectangular hyperbola are parallel to the asymptotes of another, and the centre of each lies on the other; prove that an infinity of circles can be drawn through the centre of either so as to meet the other again in conjugate triads with respect to the former.

Solution by G. F. Walker, M.A.; E. Haigh, B.A.; *and others.*

The equations to the two hyperbolas may be written

$$x^2 - y^2 = a^2, \quad 2\left[xy - ay\sec\theta - ax\tan\theta\right] = 0,$$

and, taking λ so that $\lambda(x^2-y^2-a^2) + 2\left[xy - ay\sec\theta - ax\tan\theta\right] = 0$ may represent two straight lines, we have

$$\lambda^3a^2 + 2a^2\tan\phi\sec\phi - \lambda a^2\sec^2\phi + \lambda a^2\tan^2\phi + \lambda a^2 = 0,$$

or

$$\lambda^3a^2 + 2a^2\tan\phi\sec\phi = 0.$$

Hence, for these two conics, the invariants Θ and Θ' both vanish, and conjugate triads with respect to either can be found on the other. But the circle round a conjugate triad with respect to a rectangular hyperbola passes through the centre, therefore, &c.

6829. (By Professor Wolstenholme, M.A.)—Given a fixed point S and a fixed straight line, SX is drawn perpendicular to the line and bisected in O; p being any point in the plane, pk is let fall perpendicular on the line, and kO, pS meet in P. Prove that (1) the relation between p, P is reciprocal; (2) whatever curve be traced out by p, the locus of P will be a curve of the same order and class; (3) the tangents at p, P to their loci intersect always on the fixed straight line; (4) the loci of p, P will coincide when it is a conic of which S and the given straight line are focus and corresponding directrix.

Solution by Prof. Genese, M.A.; T. Woodcock, B.A.; *and others.*

1. Let Sp meet Xk at R. Since $SO = OX$, the pencil $k(\dot{X}O\dot{S}p)$ is harmonic, hence the range $(\dot{R}P\dot{S}p)$ is so. Thus, the relation between p and P is reciprocal.

2. Let P_1P_2, p_1p_2 meet in Q. Then $Q(\dot{R}_1P_1\dot{S}p_1)$ is harmonic. Therefore to every point of the locus of p on Qp, will correspond a point on QP_2, and *vice versa;* the degree of the loci of p and P must be the same. Also, since to any line through p_1 there corresponds only one straight line through P_1, and *vice versa*, the number of tangents from p_1 to the locus of p must equal the number of tangents from P_1 to the locus of P, *i.e.*, the loci are of the same class.

3. Let p_1, p_2 be two positions of p; P_1, P_2, R_1, R_2 corresponding positions of P and R. Since $(\dot{R}_1P_1\dot{S}p_1) = (\dot{R}_2P_2\dot{S}p_2)$, therefore R_1R_2, P_1P_2, p_1p_2 cointersect; that is, the join of two points meets the join of the corresponding points on the given straight line. If p_1, p_2 be consecutive points on the locus of p, we have the theorem (3); and (4) is evident from (1). The correspondence can be exhibited very clearly by means of biangular coordinates. Since SXR is a right angle, XS bisects angle pSP. Thus, X, S being the poles, if ($\alpha = \cot SXp$, $\beta = \cot pSX$) be the coordinates of p, x, and (x, y) the coordinates of P, then $x = -\alpha$, $y = \beta$.

[More generally, the loci will coincide when it is a conic in which S and the given straight line are pole and polar.]

6804. (By the Rev. W. A. Whitworth, M.A.)—The sides of a triangle are expressed by integers in arithmetical progression; if each side be increased by 50, the radius of the inscribed circle will be increased by 17; but, if each side be increased by 60, the radius of the inscribed circle will be increased by 20; find the triangle.

Solution by Kate Gale; E Buck, B.A.; *and others.*

Let the sides be $a-b,\ a,\ a+b$; then $r = (2\sqrt{3})^{-1}\,(a^2-4b^2)^{\frac{1}{2}}$,

therefore $\qquad r+17 = (2\sqrt{3})^{-1}\left\{(a+50)^2-4b^2\right\}^{\frac{1}{2}}$,

similarly $\qquad r+20 = (2\sqrt{3})^{-1}\left\{(a+60)^2-4b^2\right\}^{\frac{1}{2}}$;

and from these equations we get $\ a = 26,\ b = 11,\ r = 4.$

6112. (By F. Morley, B.A.)—The lengths of parallel straight lines intercepted between the angles of a tetrahedron and the opposite faces are p, q, r, s, reckoned of different signs as they fall on a face or a face produced; prove that $\dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{r}+\dfrac{1}{s} = 0$, and deduce the corresponding theorem for a triangle.

Solution by G. Turriff, M.A.; E. W. Symons, B.A.; *and others.*

Let OABC be the tetrahedron, and let OD, AE, BF, CG be drawn parallel to one another through the vertices, meeting the opposite faces or these produced in D, E, F, G. If OD meet the face ABC, the other lines meet the faces produced. Take OA, OB, OC for oblique axes, and let the equations of ABC and OD be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1, \qquad \frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu} = r.$$

If OD $= p$, we get $\qquad \dfrac{\lambda}{a}+\dfrac{\mu}{b}+\dfrac{\nu}{c} = \dfrac{1}{p}$,

and evidently $\dfrac{1}{\text{AE}} = \dfrac{1}{q} = -\dfrac{\lambda}{a},\quad \dfrac{1}{\text{BF}} = \dfrac{1}{r} = -\dfrac{\mu}{b}$ and $\dfrac{1}{\text{CG}} = \dfrac{1}{s} = -\dfrac{\nu}{c}$,

therefore $\qquad \dfrac{1}{p}+\dfrac{1}{q}+\dfrac{1}{r}+\dfrac{1}{s} = 0.$

If the lines are drawn parallel to a face, one line becomes infinite and its reciprocal vanishes, and we have the case of the triangle.

6816. (By Dr. Macfarlane, F.R.S.E.)—A colony is formed by N persons of different races, and the two sexes are equally numerous. The colony remains unaffected by either immigration or emigration. What is the least time in which it can become homogeneous?

Solution by the PROPOSER.

Let p_1 denote the probability of a person dying childless; p_2 the probability of a person having children but no grandchildren; p_3 of having grandchildren but no great-grandchildren; and so on. Let k denote the average number of children for a person who has children. In order that the colony may become homogeneous in the least time, there must be no intermarriages between kindred. Under that condition, $p_2 = p_1^k, p_3 = p_1^{k^2}$, and generally $p_n = p_1^{k^n}$. Also, a person of the nth generation will have 2^n ancestors all different; hence for homogeneity we have

$$2^n = \mathrm{N}\left\{1-p_1-p_1^k-p_1^{k^2}-\ldots-p_1^{k^n}\right\}.$$

[Mr. WHITWORTH remarks that the least time is zero, the homogeneity being effected by $\mathrm{N}-1$ of the persons instantaneously dying.]

6284. (By Professor COCHEZ.)—Soit $f(x)=0$ une équation algébrique, x une quelconque de ses racines. Dans l'équation $\phi(y)=0$ dont les racines ont avec x la relation $y=f'(x)$, (1) le terme du premier degré manque; (2) le terme tout connu est le dernier terme de l'équation aux carrés des differences.

Solution by G. TURRIFF, M.A.; W. J. C. SHARP, M.A.; *and others.*

If $f(x) = (a, b, c, d \ldots \between x, 1)^n$, $f'(x) = n(a, b, c, d \ldots \between x, 1)^{n-1} = y$, and the equation in y is the eliminant of

$$(a, b, c, d \ldots \between x, 1)^{n-1} = \frac{y}{n}, \text{ and } (a, b, c, d \ldots)(x, 1)^n = 0;$$

hence the absolute term, the result of putting $y = 0$, is the discriminant,

and $$\frac{1}{n}\Sigma f'(a) = as_{n-1}+(n-1)\,bs_{n-2}+\&c. = 0,$$

which proves that the second term vanishes.

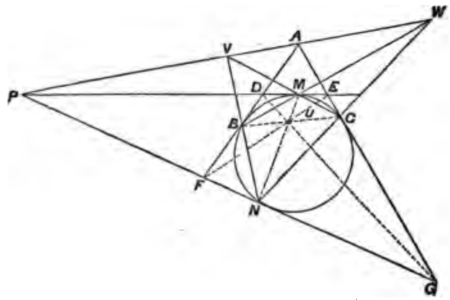
6179. (By R. KNOWLES, B.A., L.C.P.)—If a point P be taken on the asymptote to an hyperbola whose centre is C and focus S, and if PT be drawn a tangent to the curve, prove that $\angle$TSP = SPC.

Solution by A. L. SELBY, M.A.; E. RUTTER; *and others.*

The other tangent from P is the asymptote, and the line to the point of contact is therefore parallel to the asymptote; and since the tangents subtend equal angles at the focus, therefore $\angle$PST = angle between SP and line parallel to asymptote = alternate $\angle$SPC.

6720. (By T. C. Simmons, M.A.)—AB, AC are two fixed tangents to a circle [or a conic]; DE, FG are two other tangents meeting AB in D, F, and AC in E, G; prove that the point of intersection of EF and DG lies in BC.

I. *Solution by* S. Constable, M.A.; Belle Easton; *and others.*

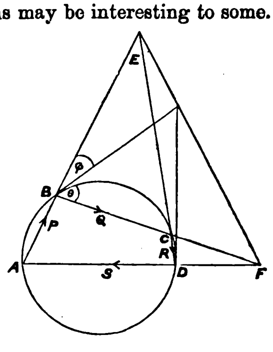


Let M, N be the points of contact of tangents DE, FG; let WV be the third diagonal of quadrilateral BNCM, and U the point of intersection of diagonals MN, BC. Then U is the pole of VW. Again, since the line joining the poles of two lines is the polar of their point of intersection, the line FE is the polar of V.

Similarly, DG is the polar of W; whence the point of intersection of FE and DG must be the pole of VW. But U is the pole of VW, therefore the lines DG, FE intersect in U, *i.e.*, on BC.

II. *Solution by* Professor Wolstenholme, M.A.

A proof of this from statical considerations may be interesting to some. Suppose forces P, Q, R, S to act along the chords AB, BC, CD, DA of a circle, and each to be inversely proportional to the chord along which it acts; let AB, CD meet in E, and AD, BC in F. Then the resultant of P, R acts through the point E, because both forces do so; and through the point F, because, the triangles FCD, FAB being similar, if p_1, p_2 be their altitudes, $\dfrac{p_1}{CD} = \dfrac{p_2}{BA}$, or $p_1 R = p_2 P$, or the moments about F are equal and opposite. The resultant therefore acts along the straight line EF. Similarly, the resultant of Q, S also acts along EF, and therefore the resultant of the four forces also does so. But, if θ, ϕ be the angles which the tangent at B makes with BC, BE, we have

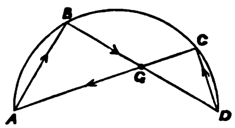


$$\frac{BC}{AB} = \frac{\sin\theta}{\sin\phi} = \frac{P}{Q};$$

whence the resultant of P, Q acts along the tangent at B. Similarly, the resultant of R, S acts along the tangent at C; and the resultant of P, Q, R, S must act through the point of intersection of the tangents at B, D.

This intersection is then a point on EF, and similarly, the tangents at A, C intersect on E, F. Thus, if a quadrilateral be formed by four tangents to a circle, a diagonal passes through two of the vertices of the quadrangle determined by the points of contact.

If we had taken the forces along AB, BD, DC, CA, and inversely proportional to these lengths, we should prove in the same way that the resultant of the four forces acts along EG, and also through the intersections of the tangents at B, C, and of the tangents at A, D.



The third case, where the resultant passes through F, G, and the intersections of the tangents at A, B, and of the tangents at C, D, will be proved by taking forces along AC, CB, BD, DA, inversely proportional to these lengths.

The general case, then, is, if we have four tangents (a, b, c, d) to a circle, the points of contact being A, B, C, D; the points (BC, AD), (AB, CD), tangents b, d, tangents a, c, lie on one straight line; the points (CA, BD), (BC, AD), tangents a, b, tangents c, d, lie on one straight line; and the points (AB, CD), (CA, BD), tangents b, c, tangents a, d, lie on one straight line. Of course the property is projective, and equally true for any conic.

6078. (By EDWYN ANTHONY, M.A.)—Prove that (1) the principal normal at any point of a helix passes through and is perpendicular to the axis of the cylinder on which the helix is traced; (2) the angle between the binormal and the axis equals the complement of the angle between the axis and the tangent.

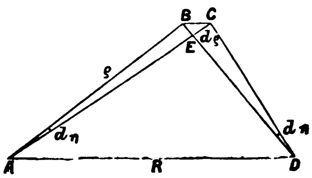
Solution by BELLE EASTON ; D. EDWARDES; *and others.*

1. The equations of the helix being $x=a\cos\theta$, $y=a\sin\theta$, $z=na\theta$, the equation of osculating plane is $n(\xi y-\eta x)+a(\zeta-z)=0$, which is satisfied by $x=0$, $y=0$, $\zeta=z$, and $\frac{dz}{ds}=\sin a$, $\therefore \frac{d^2z}{ds^2}=0$, which proves (1).

2. If ϕ be the angle which the binormal makes with axis, and a the pitch of the screw, we have $\cos\phi = a\frac{d\theta}{ds}=\frac{1}{(1+n^2)^{\frac{1}{2}}}=\cos a$.

6807. (By G. F. WALKER, M.A.)—In any curve in which the difference between the radii of absolute and spherical curvature is constant, prove that the arcs of the loci of the centres of absolute and spherical curvature (measured between corresponding points) are equal.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.



Let us denote the elements of the curves described by the centre of absolute curvature and by the centre of spherical curvature, respectively, by ds and dS, the radius of absolute curvature by ρ, the radius of spherical curvature by R, and the angle of torsion by $d\eta$. Three consecutive elements of the curve having been taken, suppose the plane of the paper to be perpendicular to the second element at its middle point A, then B and C, the centres of absolute curvature, corresponding to the osculating planes containing the first and second elements, and the second and third, respectively, will be in the plane of the paper, as will also be the two consecutive *binormals*, or corresponding axes of osculating planes through B and C, intersecting in D, the centre of spherical curvature. The angles BAC and BDC will each be the angle of torsion $d\eta$, EC will be $d\rho$, and CD (or *ultimately* BD) will be $\frac{d\rho}{d\eta}$. As the consecutive centre of spherical curvature lies on CD, CD is a tangent to the locus of the centre of spherical curvature, and by a well-known theorem (WILLIAMSON'S *Diff. Calc.*, 4th ed., p. 233), we have $d\,(\mathrm{BD}) = d\mathrm{S} - \mathrm{BE}$, or $d\mathrm{S} = \rho\, d\eta + d\,\frac{d\rho}{d\eta}$; but, as $\mathrm{R} - \rho = \text{const.}$, $(\mathrm{AB}^2 + \mathrm{BD}^2)^{\frac12} - \rho = \text{const.}$, or $\left(\rho^2 + \frac{d\rho^2}{d\eta^2}\right)^{\frac12} - \rho = \text{const.}$; or, differentiating, $\rho\, d\eta + d\,\frac{d}{d\eta} = (\rho^2 d\eta^2 + d\rho^2)^{\frac12} = (\mathrm{BE}^2 + \mathrm{EC}^2)^{\frac12} = \mathrm{BC} = ds$; therefore $d\mathrm{S} = ds$, or $\mathrm{S}_1 - \mathrm{S}_0 = s_1 - s_0$.

More generally, if R and ρ be connected by the relation $\mathrm{R} - k\rho = c$, where k and c are both constants, including the case in which c is zero, and, consequently, the ratio of R to ρ is given, it follows nearly as above that $\mathrm{S}_1 - \mathrm{S}_0 = k\,(s_1 - s_0)$.

[The radius of curvature of the edge $\left(\frac{d\mathrm{S}}{d\mathrm{T}}\right) = \frac{\mathrm{R}d\mathrm{R}}{d\rho}$, and $d\mathrm{S} = \mathrm{R}d\mathrm{T}$; therefore, since $d\mathrm{R} = d\rho$, we see that $ds = \mathrm{R}d\mathrm{T} = ds\theta$, and $\mathrm{S} = s$ between corresponding points.]

6398. (By R. E. RILEY, B.A.)—Prove that if

$$(a+b+c)^3 = a^3 + b^3 + c^3, \text{ then } (a+b+c)^{2n+1} = a^{2n+1} + b^{2n+1} + c^{2n+1}.$$

Solution by E. W. SYMONS, M.A.; G. TURRIFF, M.A.; *and others.*

We have $(a+b+c)^3 \equiv a^3 + b^3 + c^3 + 3\,(b+c)(c+a)(a+b) = a^3 + b^3 + c^3$,

therefore $\qquad (b+c)(c+a)(a+b) = 0;$

and, putting any one of these factors $= 0$, the required result follows.

6270. (By J. L. McKenzie, B.A.)—**Express in terms of the coefficients of two quadratic equations, the condition that one root of the first should have a given ratio to one root of the second.**

Solution by W. J. Constable, B.A.; J. O'Regan; *and others.*

Let the equations be $a_1x^2+b_1x+c_1=0$, $a_2x^2+b_2x+c_2=0$, and the ratio $m:1$; then, eliminating x from

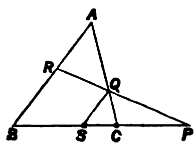
$$a_1x^2+b_1x+c_1=0,\quad a_2m^2x^2+b_2mx+c_2=0,$$

the required condition is

$$(b_1c_2-b_2c_1m)(b_1a_2m^2-a_1b_2m)=(c_1a_2m^2-a_1c_2)^2.$$

6092. (By J. Young, B.A.)—**Through a point in the base of a triangle produced draw a straight line cutting the sides so that the rectangle contained by the segment of one side towards the base and the segment of the other towards the vertex of the triangle shall be a maximum.**

Solution by G. Heppel, M.A.; Christine Ladd; *and others.*



Drawing QS parallel to AB, let PC $=h$, PB $=k$, PS $=x$, then $k-h=a$,

$$\text{QC}=\frac{b}{a}(x-h),\quad \text{AR}=c-\frac{kc}{ax}(x-h);$$

$$\text{AR}\,.\,\text{QC}=\frac{bc}{a}\left\{x-h-\frac{k}{a}\left(x-2h+\frac{h^2}{x}\right)\right\}$$

$$=\text{a maximum when } 1-\frac{k}{a}+\frac{kh^2}{ax^2}=0,$$

that is, when $\quad (k-a)x^2=kh^2,\ \text{ or }\ x^2=kh,$

so that then PS is a mean proportional between PB and PC.

Exactly the same result is obtained if the rectangle be AQ . BR.

6800. (By the Editor.)—Prove that, if

$$\frac{ayz}{y^2+z^2}=\frac{bzx}{z^2+x^2}=\frac{cxy}{x^2+y^2}=1,\ \text{then}\ a^2+b^2+c^2=abc+4.$$

Solution by Arthur Cohen, Q.C., M.P.; E. Rutter; *and others.*

Let $l=\dfrac{y}{z},\ m=\dfrac{z}{x},\ n=\dfrac{x}{y}$; then we have $lmn=1$,

$$a = l+mn, \quad b = m+nl, \quad c = n+lm, \quad \Sigma a^2 = \Sigma l^2 + \Sigma m^2n^2 + 6;$$

$$abc = \frac{1}{lmn}\left\{(lmn+l^2)(lmn+m^2)(lmn+n^2)\right\}.$$

$$= (1+l^2)(1+m^2)(1+n^2) = 1+\Sigma l^2 + \Sigma m^2n^2 + 1 = \Sigma a^2 - 4.$$

[Otherwise:—$(a^2+b^2+c^2)\,x^2y^2z^2 = x^2(y^2+z^2)^2 + y^2(z^2+x^2)^2 + z^2(x^2+y^2)^2$
$\equiv (x^2+y^2)(y^2+z^2)(z^2+x^2) + 4x^2y^2z^2 = abc\,x^2y^2z^2 + 4x^2y^2z^2$; therefore &c.]

NOTE ON QUESTION 6800; *by* Professor CAYLEY, F.R.S.

The identity given by the solution is a very interesting one. Instead of a, b, c, writing $(a, b, c) \div d$, we have $4d^3 - d(a^2+b^2+c^2) + abc = 0$, satisfied by $a : b : c : d = x(y^2+z^2) : y(z^2+x^2) : z(x^2+y^2) : xyz$;

or, considering (a, b, c, d) as the coordinates of a point in space, and (x, y, z) as the coordinates of a point in a plane, we have thus a correspondence between the points of the cubic surface $4d^3 - d(a^2+b^2+c^2) + abc = 0$, and the points of the plane. To a given system of values of (x, y, z) there corresponds, it is clear, a single system of values of (a, b, c, d); and it may be shown without difficulty that to a given system of values of (a, b, c, d) satisfying the equation of the surface there correspond two systems of values of (x, y, z); the plane and cubic surface have thus a (1, 2) correspondence with each other.

6432. (By J. HAMMOND, M.A.)—Prove that $PG = 2\rho$ if PG be the normal measured from the curve to the axis of x, and ρ the radius of curvature of the curve

$$x = \frac{c}{2^{\frac{1}{2}}}\int\frac{d\theta}{(1-\frac{1}{2}\sin^2\theta)^{\frac{1}{2}}} - c\sqrt{2}\int(1-\tfrac{1}{2}\sin^2\theta)^{\frac{1}{2}}\,d\theta, \quad y = c\cos\theta.$$

Solution by C. MORGAN, B.A.; D. EDWARDES; *and others.*

The property $PG = 2\rho$ leads to the equation

$$py\frac{dp}{dy} + 2p^2 + 2 = 0, \ \left(p \equiv \frac{dy}{dx}\right), \quad \text{whence } 1+p^2 = \frac{c^4}{y^4}.$$

Putting $y = c\cos\theta$, this gives $\dfrac{dx}{d\theta} = \dfrac{c\cos^2\theta}{(1+\cos^2\theta)}$,

therefore $\quad x = c\sqrt{2}\displaystyle\int(1-\tfrac{1}{2}\sin^2\theta)^{\frac{1}{2}}\,d\theta - \frac{c}{2^{\frac{1}{2}}}\int\frac{d\theta}{(1-\frac{1}{2}\sin^2\theta)^{\frac{1}{2}}}$.

6289. (By G. J. GRIFFITHS, M.A.)—A uniform heavy string of length $2a$ is placed on a smooth cardioid $r = a(1+\cos\theta)$, whose axis is horizontal; one end of the string being at the apse; and the string is allowed to run off the curve; prove that its velocity v when just leaving the curve is given by the equation $v^2 = \frac{1}{10}ga(52-3^{\frac{5}{2}})$.

Solution by D. EDWARDES; J. O'REGAN; *and others.*

If θ be the vectorial angle of a point on the curve, and ϕ the inclination of the tangent at that point to the vertical, $\phi = \frac{3}{2}\theta$. Let z be the length of string which is vertical at the time t. Then, equating the rate of increase of momentum to the impressed forces, we have

$$2av\frac{dv}{dz} = gz + g\int_0^{2a-z}\cos\phi\,ds;$$

the mass of a unit length of string being taken as the unit of mass, and s is measured from the apse.

Now $s = 4a\sin\frac{1}{2}\theta$, $\therefore \int\cos\phi\,ds = 4a\sin\frac{1}{2}\theta\cos^3\frac{1}{2}\theta = s\left(1-\frac{s^2}{16a^2}\right)^{\frac{3}{2}}$.

Hence, integrating, and putting $2a-z = s'$, we get

$$av^2 = \tfrac{1}{2}gz^2 - g\int s'\left(1-\frac{s'^2}{16a^2}\right)^{\frac{3}{2}}ds' + \mathrm{C} = \tfrac{1}{2}gz^2 + \tfrac{16}{5}a^2g\left(1-\frac{s'^2}{16a^2}\right)^{\frac{5}{2}} + \mathrm{C},$$

and $v = 0$ when $s' = 2a$, therefore $\mathrm{C} = -\frac{3^{\frac{5}{2}}}{10}a^2g$. Hence, &c., putting $s' = 0$.

6649. (By J. O'REGAN.)—If a quadrilateral be circumscribed to a circle and a fifth variable tangent be drawn, the rectangles under perpendiculars on it from each pair of opposite angles are in a constant ratio.

Solution by G. HEPPEL, M.A.; Rev. T. R. TERRY, F.R.A.S.; *and others.*

Taking the centre of the circle as origin, and representing the vectorial angles of the points of contact by α, β, γ, δ, the equations to the four tangents are $\qquad x\cos\alpha + y\sin\alpha - a = 0$, &c.

The coordinates of the intersection of the first two are

$$\frac{a\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)},\quad \frac{a\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}.$$

The length of the perpendicular from this point on the variable tangent

$x\cos\theta + y\sin\theta - a = 0$, is $2a\sin\frac{1}{2}(\theta-\alpha)\sin\frac{1}{2}(\beta-\theta)\sec\frac{1}{2}(\alpha-\beta)$;

hence the ratio of rectangles under the perpendiculars is

$$\frac{\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\delta-\alpha)}{\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\gamma-\delta)}.$$

6765. (By Prince CAMILLE DE POLIGNAC.)—Find the condition for the convergence of the continued fraction $A + \frac{p}{q+}\,\frac{p}{q+}\,\frac{p}{q+}$ &c.

Solution by the PROPOSER.

Starting with $x_n = \dfrac{Ax_{n-1}+B}{A'x_{n-1}+B'}$, and transforming this expression as follows,

$$A'x_n = \frac{A(A'x_{n-1}+B')+BA'-AB'}{A'x_{n-1}+B'} = A + \frac{BA'-AB'}{A'x_{n-1}+B'},$$

$$A'x_n + B' = A + B' + \frac{BA'-AB'}{A'x_{n-1}+B'};$$

putting $BA'-AB' = p$, $A+B' = q$, we have

$$A'x_n + B' = q + \frac{p}{A'x_{n-1}+B'}, \quad \text{therefore } A'x_n = A + \frac{p}{q+}\,\frac{p}{q+}\ldots\frac{p}{A'x_0+B'},$$

where p occurs n times.

This expansion gives, to a factor, the n^{th} power of the linear substitution we started with. Now we know, from the solution to Quest. 6735,* where these powers have been represented as homographic points on a line, that they converge towards the foci of the system when real.

Hence the condition of convergence is that the roots of the quadratic $A'x^2 + (B'-A)x - B = 0$ should be real, that is to say, that

$$(B'-A)^2 + 4BA' \geqq 0 \text{ or } (B'+A)^2 + 4(BA'-AB') \geqq 0, \text{ or } q^2 + 4p \geqq 0.$$

6783. (By R. KNOWLES, B.A., L.C.P.)—Prove that (1) if the axes be rectangular, the equation to the locus of the vertices of all parabolas whose chords of contact cut off a triangle of constant area $(\frac{1}{2}a^2)$, is $(x^7y^{-1})^{\frac{1}{12}} + (y^7x^{-1})^{\frac{1}{12}} = a^{\frac{1}{2}}$; (2) that of the foci is $(x^2+y^2)^2 = a^2xy$; (3) the chords of contact always touch at their middle points a rectangular hyperbola, to which the axes are asymptotes.

Solution by W. B. GROVE, B.A.; CHRISTINE LADD; *and others.*

Let the equation to the parabola be $\left(\frac{x}{h}\right)^{\frac{1}{2}} + \left(\frac{y}{k}\right)^{\frac{1}{2}} = 1$, where $hk = a^2$;

then the chord of contact and perpendicular thereon from the origin are

$$\frac{x}{h} + \frac{y}{k} = 1, \quad ky = hx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1, 2);$$

and the coordinates of the intersection of (1), (2), that is, the focus S, are $\dfrac{hk^2}{h^2+k^2}$, $\dfrac{h^2k}{h^2+k^2}$; therefore the locus of S is $(x^2+y^2)^2 = a^2xy$.

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* See pp. 93 and 94 of this volume.

Again, the equation to the line through S, making with the axis of x an angle equal to that which the chord of contact makes with the same, and the coordinates of this line's intersection with the curve, *i.e.*, the *vertex*, are

$$y-\frac{h^2k}{h^2+k}=\frac{k}{h}\left(x-\frac{hk^2}{h^2+k}\right),\quad \frac{hk^2}{(h+k)},\quad \frac{h^2k}{(h^2+k^2)^2};$$

hence the locus of the vertex is $\left(\frac{x}{y}\right)^{\frac13}+\left(\frac{y}{x}\right)^{\frac13}=\left(\frac{a^2}{xy}\right)^{\frac13}$.

Lastly, the envelop of (1), subject to the condition $hk=a^2$, is $4xy=a^2$, and the coordinates of the point of contact are $\frac12 h$, $\frac12 k$; hence the chords of contact always touch at their middle points, &c.

6572. (See p. 62). II. *Solution by* J. YOUNG, B.A.; M. BAKER, M.A.; *and others.*

In the diagram to the following Solution, let the inner circle touch AB in F, and let the circle through AOB cut OF produced in G; then, since $OD = AD - OF$, we have $OD^2 = AD^2 + OF^2 - AB \cdot OF = OF^2 + FD^2$;

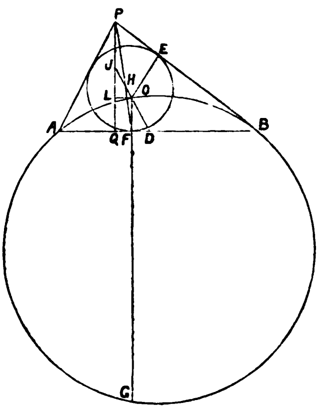
therefore $\quad AB \cdot OF = AD^2 - FD^2 = AF \cdot FB = OF \cdot FG$;

hence $AB = FG$ = the radius of the circle touching AB and the lines PA, PB produced, because the centre of this circle lies on the circle AOB, and the distance from B of its point of contact with AB is equal to AF; therefore $AB \cdot PE$ = area $PAB = \frac12 PQ \cdot AB$, hence $PE = \frac12 PQ$; therefore, &c.

The proof is similar for the external contact.

III. *Solution by* D. EDWARDES; Prof. MATZ, M.A.; *and others.*

It may be shown that the line DO, which joins the centre of the inscribed circle to the middle point of the base, bisects in H the line PF joining the vertex to the point where the inscribed circle touches the base. Writing then for shortness p, r, c, t for PQ, OF, AB, PE respectively, putting J at the intersection of OD and PQ, and drawing OL at right angles to PQ, we have $PJ = r$, and



$$OP^2 = PJ^2 + OJ^2 + 2PJ \cdot JL,$$

or $\quad t^2 = OJ + 2r(p-2r)$ ..... (1).

By similar triangles,

$$\frac{p-2r}{OJ}=\frac{2r}{c-2r}\ (\text{since } OD=\tfrac12 c-r).$$

But area of $PAB = \frac12 pc$

and also $\quad = r(c+t)$ ... (2),

or $\quad \frac{p-2r}{2r}=\frac{t}{c}=\frac{t-p+2r}{c-2r}$;

therefore $OJ = t-p+2r$;

and therefore, by (1),

$$t^2=(t-p+2r)^2+2r(p-2r),\ \text{whence } p=2t \ \ldots\ldots\ldots\ldots (3).$$

Hence equations (2) and (3) give $\frac{1}{r} - \frac{2}{p}$ = constant, which (since $\frac{2r}{p} = \tan \frac{1}{2}P$), is equivalent to $\tan \frac{1}{2}A + \tan \frac{1}{2}B = 1$.

[That D, O, H are in a straight line follows from the vanishing of the determinant $\begin{vmatrix} \frac{\Delta}{a}, & \frac{\Delta}{b}, & 0 \\ r, & r, & r \\ r\cos^2 \frac{1}{2}B, & r\cos^2 \frac{1}{2}A, & \frac{rs}{c} \end{vmatrix}$, whereof the rows are the trilinear coordinates of these three points respectively.]

6834. (By the EDITOR.)—Trace the locus of a point whose distance from a given line $y+c=0$ is equal to the sum (including, throughout, *difference*) of its distances from two given points (a, b), $(-a, -b)$, showing that (1) its general equation is

$$[(y+c)^2 + 4(ax+by)]^2 = 4(y+c)^2[(x+a)^2+(y+b)^2];$$

(2) when $b=0$, so that the curve is the locus of the vertex of a triangle on a given base, having the sum of its two sides equal to the sum of the perpendicular from the vertex on the base and a given line (c), the equation is $(y+c)^4 + 16a^2x^2 = 4(y+c)^2(a^2+x^2+y^2)$; (3) when $c > 2a$, the curve (2) consists of a loop and four infinite branches that cross at the intersection of the y-axis with the given line, and have an asymptote parallel to that line at a distance $2a$ on each side therefrom; (4) analogous but varying forms subsist when $2a > c > a\sqrt{3}$ and $c < a\sqrt{3}$, with a conjugate point on the y-axis when $c = a\sqrt{3}$; (5) when $c = 2a$, so that, in the triangle-locus (2), the sum of the sides is equal to the sum of the base and perpendicular, the equation is $4(a^2-x^2)(y+4a) = (3y+8a)y^2$, and then two of the infinite branches degenerate into the base of the triangle, and the other two unite continuously with the loop and cut the y-axis at an angle of $48^\circ\, 12' = \cos^{-1}\frac{2}{3}$; (6) the curve (5) is also the locus of the intersection of tangents drawn from the ends of a diameter of a circle to a circle that touches the given circle and its diameter; (7) the areas of the loop of the curve (5), of the part cut off by the base, and of the space between the infinite branches and the asymptote, are, if we put $\sin^2\beta = \frac{5}{8}$,

$$\frac{2}{3^{\frac{3}{2}}}(7\sqrt{15} - 16\beta)\,a^2, \quad \frac{16}{27}(9-\pi\sqrt{3})\,a^2, \quad \frac{2}{3^{\frac{3}{2}}}(7\sqrt{15}+16\beta)\,a^2;$$

(8) the locus of the centre of the touching circle in (6) is a parabola; and (9) the sum of the tangents of the angles subtended at each end of the given diameter in (6) by lines drawn from the other end to the centre of the touching circle, is unity. (See solution of Quest. 6572.)

Solution by D. EDWARDES; G. M. REEVES, M.A.; *and others.*

1. We have $[(x+a)^2+(y+b)^2]^{\frac{1}{2}} \pm [(x-a)^2+(y-b)^2]^{\frac{1}{2}} = y+c$,
and, since $[(x+a)^2+(y+b)^2] - [(x-a)^2+(y-b)^2] \equiv 4(ax+by)$,

therefore $[(x+a)^2+(y+b)^2]^{\frac{1}{2}} \mp [(x-a)^2+(y-b)^2]^{\frac{1}{2}} = 4\,(ax+by)\,(y+c)^{-1}$,
whence, by adding and squaring, we get the result stated.

2, 3, 4. Putting $b = 0$, we have the result stated, which is equivalent

to $$4x^2\,[(y+c)^2-4a^2] = (y+c)^2\,[(y+c)^2-4\,(y^2+a^2)],$$

or $$4x^2\,(2a-c-y)\,(2a+c+y) = 3\,(y+c)^2\,(y-a_1)\,(y-a_2),$$

where $$a_1,\ a_2 \equiv \tfrac{1}{3}\,[c \pm 2\,(c^2-3a^2)^{\frac{1}{2}}].$$

By the usual method, the asymptotes are found to be $y+c = \pm\, 2a$, as stated in the Question. When $y = 0$, $x = \pm\,\frac{1}{2}c$, and the curve cuts the x-axis at $\tan^{-1} 2$. When $y = -c$, $x = 0$, and the tangents are $(c^2+a^2)\,y^2-4a^2x^2 = 0$. a_2 is + or −, according as $c \lesseqgtr 2a$, also $a_2 \gtrless 2a-c$, according as $c \gtrless 2a$. When $c < 2a$, then, while y is positive and $< a_2$, x is $\pm$. When $y = a_2$, the curve cuts the y-axis at right angles. Also, c being less than $2a$, $a_2 < 2a-c$; hence, if $a_1 > y > a_2$, x is $\sqrt{-1}$. At $y = a_1$, the curve cuts the y-axis at right angles. When $2a-c > y > a_1$, x is $\pm$; and when $y = 2a-c$, $x = \infty$. The several cases are exhibited in the diagrams. As c increases, the asymptotes continually descend. When $c = a\sqrt{3}$, there is a conjugate point on the y-axis, where $y = \frac{1}{3}c$, and above the asymptote. Figs. 1, 2 are respectively drawn for $c = a$, $1\frac{1}{2}a$ nearly.

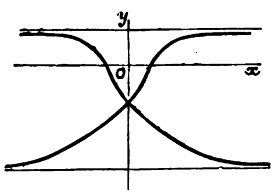


Fig. 1. $c < a\sqrt{3}$.

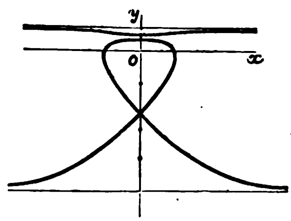


Fig. 2. $2a > c > a\sqrt{3}$.

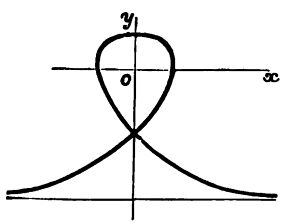


Fig. 3. $c = 2a$.

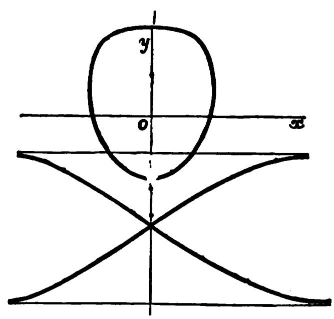


Fig. 4. $c > 2a$.

5. When $c = 2a$, the equation becomes

$$4x^2\,(y+4a) = (y+2a)^2\,(4a-3y) \text{ or } 4\,(a^2-x^2)\,(y+4a) = (3y+8a)\,y^2.$$

6. In the Fig. to Quest. 6572 (see pp. 62, 110), let $PQ = y$, $DQ = x$, $\tan \frac{1}{2}A \tan \frac{1}{2}B = z$, then, using the property $\tan \frac{1}{2}A + \tan \frac{1}{2}B = 1$, we get

$$z = \frac{4(a^2 - x^2)}{y^2} - 2, \quad \frac{2ay}{a^2 - x^2} = \frac{2 - 2z}{2z + z^2},$$

whence, eliminating z, we have the equation in (5).

7. [Our correspondents all find the areas from the equation in (5), but they may be more simply obtained by removing the origin to the vertex or top of the loop (Fig. 3), and taking the central line as positive x-axis, when the equation becomes

$$4(16a - 3x)y^2 = x(10a - 3x)^2;$$

whence, putting $3x = 16a \sin^2\theta$, the area of any part of the curve is

$$\Sigma = \int y\,dx = \int y\left(\frac{dx}{d\theta}\right)d\theta = \frac{16a^2}{3\sqrt{3}}\int(-1 + 3\cos 2\theta - 2\cos 4\theta)\,d\theta$$
$$= \tfrac{8}{9}\sqrt{3}\,(-2\theta + 3\sin 2\theta - \sin 4\theta)\,a^2.$$

The auxiliary angle θ is the angle subtended at the lower (or asymptotic) end of the vertical axis by the horizontal y-ordinate to a circle on the whole axis as diameter; and the angle β is the value of θ at the node.

The areas in the question, in order, are obtained by taking the double of Σ between the θ-limits $(0, \beta)$, $(0, \frac{1}{3}\pi)$, $(\beta, \frac{1}{3}\pi)$.]

8. The centre O of the variable circle (see Figures to solution of Question 6572, pp. 62, 110) is equidistant from D and from the tangent to the semicircle parallel to AB; hence the locus of O is a parabola.

[The general curve may be readily extended so as to give the locus of a point whose distance from a fixed straight line bears a given ratio to the sum or difference of its distances from two fixed points.]

6861. (By Professor Genese, M.A.)—A, B, C, D are fixed points. A circle is drawn through AB; then two can be drawn through CD to touch it; prove that the locus of the points of contact is a bicircular quartic.

Solution by Prof. Townsend, F.R.S.; Prof. Nash, M.A; *and others.*

More generally, A, B, C, D, E, F being fixed points in a common plane, if a conic through ABEF touch a conic through CDEF, the locus of the points of contact is a binodal quartic of which E and F are the nodal points. For, if U and V be the two conics passing through all the points except B and A respectively; then, since six conics may be described through ABEF, three touching U at E, at F, and at A respectively, and three touching V at E, at F, and at B respectively, the two points A and B, and for a similar reason the two C and D, are single, and the two E and F are double points of the locus in question; the order of which is seen to be fourth, from the consideration that the two lines AB and CD can meet it again, the former only at the two points P and Q of its contact with two

of the system of conics through CDEF, and the latter only at the two points R and S of its contact with two of the system of conics through ABEF. When the two points E and F, common to the two systems of conics, are the two I and J in the plane of the points, the property is manifestly that of the question.

When the six points A, B, C, D, E, F lie on a conic in the plane, the locus in question breaks up into the containing conic, and another intersecting it at E and F, and also at the points of contact M and N of the tangents to it from the intersection of the lines AB and CD; the pairs of tangents to the two conics at M and N dividing harmonically the angles EMF and ENF at which M and N are subtended by EF; and the two conics themselves becoming two circles intersecting at right angles when E and F coincide with I and J, as in the question.

[Inverting with respect to C, we have to find the locus of the points of contact of tangents from a fixed point D to a circle passing through two fixed points. If a be constant and α variable, the equations of circle and polar of (X, Y) are

$$x^2+y^2-a^2 = 2\alpha y, \quad x\mathrm{X}+y\mathrm{Y}-a^2 = \alpha(y+\mathrm{Y}) \qquad\text{............(1, 2)};$$

and locus of intersection of (1), (2) is

$$(x^2+y^2-a^2)(y+\mathrm{Y}) = 2y\,(x\mathrm{X}+y\mathrm{Y}-a^2),$$

a circular cubic, the inverse of which is a bicircular quartic, passing through the four points A, B, C, D.]

NOTE ON QUESTION 6816. *By the* EDITOR.

In reference to the solution given on pp. 101, 102, Mr. WHITWORTH states that he does "not quite see what Dr. MACFARLANE means, but his solution, I think, must be wrong because 'least time' cannot be a function of 'averages' or 'probabilities.' To calculate the least time in which a result can be brought about, we have nothing whatever to do with the *chances* of the causes which bring about that result, but each must be assumed to happen in the way most favourable for the result. I am quite serious in offering, as the only true solution of the question, that which has been bracketed as a remark at the foot of the solution."

To this Dr. MACFARLANE replies, that he "cannot altogether agree with the principle which Mr. WHITWORTH lays down; for, may we not consider all the causes but one to act in an average manner, and investigate what follows from supposing the remaining cause to act in the most favourable way? It is assumed, in my solution, that the colony has an average experience in every respect, excepting that they adopt the law of marriage which is most favourable for producing homogeneity. The minimum refers to this last variable. Mr. WHITWORTH's may be the absolute minimum, but mine is a relative, and I think the relevant, minimum. By the term homogeneous is meant, descended from a common group of ancestors, with whatever that fact may imply. If N $(1-p_1-p_1^k-\&c.)$ is not an exact power of 2, the number next above, which is a power of 2, is to be taken."

6092 (See p. 106). II. *Solution by the* PROPOSER.

The triangle CQR (see Fig. to former solution) is a maximum when the vertex S of the parallelogram ARQ lies on BC (MULCAHY's *Geometry*, p. 88); and in this position of S, PC . PS = PQ : PR = PS : PB, or PS is a mean proportional between PC and PB. Draw RT perpendicular to AC: then the triangle CQR = ½CQ . RT, and RT is constantly proportional to AR, therefore, &c. Since CQ . AR : AQ . BR = CP : PB, the rectangles CQ . AR and AQ . BR are a maximum at the same time.

6882. (By BELLE EASTON.)—Through a point P, between two lines AB, AC given in position, draw a line such that the rectangle under the parts thereof between the point and those lines may be a minimum.

Solution by ARTHUR COHEN, Q.C., M.P.; J. O'REGAN; *and others.*

Let qPr be the required line, so that $Pq \cdot Pr$ is a minimum; and let $q'Pr'$ be a contiguous line; then it is evident that ultimately $Pq \cdot Pr$ must equal $Pq' \cdot Pr'$; therefore the points q, q', r, r' must lie on a circle, and AB, AC must be tangents to the circle at q and r respectively; therefore $Aq = Ar$. Draw then AD bisecting the angle A, and Pqr perpendicular on it; then Pqr is the required line.

PROOF THAT AN EQUATION MUST HAVE AT LEAST n ROOTS.
By J. HAMMOND, M.A.

The roots of the equation $x^n + ax^{n-1} + bx^{n-2} + \ldots + kx + l = 0$, if any such exist, can only be functions of n and of the coefficients; *i.e.*, they must be of the form $\phi(n, a, b, c, \ldots l)$. The roots of any special equation will be the results of substituting special values for $a, b, c, \ldots l$ in each of the functions ϕ. Now, if there were only m of these functions, we could not have *more* than m special values of them, all distinct from one another, though we might have *less* than m *distinct* values of the functions in some special cases. Or, the number of *distinct* roots of a special equation *cannot be more, though it may be less*, than the number of roots of the general equation. Now, in the special case $x^n = 1$, we know of n distinct roots; therefore n cannot be greater, though it may be less, than the number of roots of the general equation of the n^{th} degree; or, the general equation of the n^{th} degree cannot have less than n roots.

6778. (By J. J. WALKER, M.A.)—Show that the work done (in *C.G.'s*) in raising the piston of a suction pump, so as to elevate the water b *cms* in

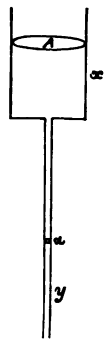
the suction-tube (of section α) from a point a *cms* below the bottom of the working-barrel, is equal to

$$\alpha\left\{\tfrac{1}{2}b^2-a\mathrm{H}\log_e\left[\mathrm{H}+(\mathrm{H}-b)\right]+\mathrm{H}ab+(\mathrm{H}-b)\right\},$$

H being the height of the water-barometer; and explain the significance of the separate terms.

Solution by G. F. WALKER, M.A.; G. M. REEVES, M.A.; *and others.*

Work in *C.G.'s* is $\int\frac{\mathrm{A}dx\,(\Pi-p)}{g\rho}$, where A is area of piston and x the height through which it is raised.



Let y be height to which water rises.

Then $p+g\rho y=\Pi$, and $p\left[\mathrm{A}x+\alpha(a-y)\right]=\pi\alpha a$;

$$\mathrm{A}x+\alpha(a-y)=\frac{a\alpha\mathrm{H}}{\mathrm{H}-y},$$

$$\mathrm{A}dx=\left[\alpha+\frac{a\alpha\mathrm{H}}{(\mathrm{H}-y)^2}\right]dy;$$

$$\text{work}=\int_0^b y\left[\alpha+\frac{a\alpha\mathrm{H}}{(\mathrm{H}-y)^2}\right]dy,$$

piston raised slowly,

$$=\alpha\left[\tfrac{1}{2}b^2+a\mathrm{H}^2\int_0^b\frac{dy}{(\mathrm{H}-y)^2}-a\mathrm{H}\int_0^b\frac{dy}{\mathrm{H}-y}\right]$$

$$=\alpha\left[\tfrac{1}{2}b^2+\frac{ab\mathrm{H}}{\mathrm{H}-b}-a\mathrm{H}\log\frac{\mathrm{H}}{\mathrm{H}-b}\right].$$

[The first term is the work done in raising a column of water of height b, irrespective of atmospheric pressure; the second is the work done against external pressure; the third, that done by internal pressure, in the process of expanding the column of air occupying the suction tube from pressure H to pressure H $-\,b$.]

6859. (By Professor SIMON NEWCOMB, M.A.)—Prove that

$$\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right)=-\eta^2+\tfrac{1}{2}\eta^4-\tfrac{1}{3}\eta^6+\ldots-2\eta\cos x$$

$$-\tfrac{1}{2}\,.\,2\eta^2\cos 2x-\tfrac{1}{3}\,.\,2\eta^3\cos 3x-\ldots=\Sigma_{i=1}^{i=\infty}(-1)^i\frac{\eta^{2i}}{i}-\Sigma_{i=1}^{i=\infty}\frac{2\eta^i}{i}\cos ix.$$

Solution by DR. MACFARLANE, F.R.S.E.

In the *Mathematical Visitor* for 1880, I proposed an expansion, the proof of which involves that of Professor NEWCOMB. It is

$$\log\left[a^2+b^2+2ab\cos(\alpha-\beta)\right]^{\frac{1}{2}}=\log a+\frac{b}{a}\cos(\alpha-\beta)-\tfrac{1}{2}\frac{b^2}{a^2}\cos 2(\alpha-\beta)+\text{etc.}$$

I was led to this expansion by the following considerations:—

Let a denote a vector of length a and angle α, and b a vector of length b and angle β, and so on, the vectors being restricted to lying in one plane; then, as the combinations of such quantities are subject to all the laws of the ordinary algebra, all the known theorems are true when we substitute such directed quantities for the ordinary linear quantities. Thus, for the particular example, we have

$$\log(\mathrm{a}+\mathrm{b}) = \log \mathrm{a} + \log\left(1+\frac{\mathrm{b}}{\mathrm{a}}\right). \quad \text{Now, } \log \mathrm{a} = \log a + i \log \alpha,$$

$$\log\left(1+\frac{\mathrm{b}}{\mathrm{a}}\right) = \frac{\mathrm{b}}{\mathrm{a}} - \tfrac{1}{2}\left(\frac{\mathrm{b}}{\mathrm{a}}\right)^2 + \tfrac{1}{3}\left(\frac{\mathrm{b}}{\mathrm{a}}\right)^3 - \dots$$

$$= \frac{b}{a}\cos(\beta-\alpha) - \tfrac{1}{2}\left(\frac{b}{a}\right)^2 \cos 2(\beta-\alpha) + \dots$$

$$+ i\left\{\frac{b}{a}\sin(\beta-\alpha) - \tfrac{1}{2}\left(\frac{b}{a}\right)^2 \sin 2(\beta-\alpha) + \dots\right\}.$$

Also $\log(\mathrm{a}+\mathrm{b}) = \log\left\{[a^2+b^2+2ab\cos(\beta-\alpha)]^{\frac{1}{2}} . \tan^{-1}\dfrac{a\sin\alpha + b\sin\beta}{a\cos\alpha + b\cos\beta}\right\}$,

where the dot separates the tensor and versor parts,

$$= \tfrac{1}{2}\log[a^2+b^2+2ab\cos(\beta-\alpha)] + i\log\tan^{-1}\frac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta};$$

therefore $\log[a^2+b^2+2ab\cos(\beta-\alpha)]$

$$= 2\log a + 2\left\{\frac{b}{a}\cos(\beta-\alpha) - \tfrac{1}{2}\left(\frac{b}{a}\right)^2\cos 2(\beta-\alpha) + \text{etc.}\right\} \quad \text{......(1)},$$

and $\log\tan^{-1}\dfrac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}$

$$= \log\alpha + \frac{b}{a}\sin(\beta-\alpha) - \tfrac{1}{2}\left(\frac{b}{a}\right)^2 \sin 2(\beta-\alpha) + \dots \quad \text{.........(2)}.$$

Let $a^2+b^2 = 1$, $ab = \dfrac{-\eta}{1+\eta^2}$, and $\beta-\alpha = x$, then equation (1) becomes the expansion required by Prof. NEWCOMB. Equation (2) gives the complementary expansion.

The expansion may also be proved as follows:—

$$\log\left(1-\frac{2\eta}{1+\eta^2}\cos x\right) = -\log(1+\eta^2) + \log[1-\eta(\epsilon^{x\sqrt{-1}}+\epsilon^{-x\sqrt{-1}}+\eta^2),$$

$$= -\log(1+\eta^2) + \log(1-\eta\epsilon^{x\sqrt{-1}}) + \log(1-\eta\epsilon^{-x\sqrt{-1}}),$$

$$= -\eta^2 + \tfrac{1}{2}\eta^4 - \text{etc.} - \eta(\epsilon^{x\sqrt{-1}}+\epsilon^{-x\sqrt{-1}}) - \tfrac{1}{2}\eta^2(\epsilon^{2x\sqrt{-1}}+\epsilon^{-2x\sqrt{-1}}) - \text{etc.}$$

$$= -\eta^2 + \tfrac{1}{2}\eta^4 - \text{etc.} - 2(\eta\cos x + \tfrac{1}{2}\eta^2\cos 2x + \text{etc.}).$$

6826. (See p. 97). II. *Solution by* W. W. TAYLOR, M.A.

The former solution of this problem contains the following curious error:—It is at first correctly proved that every measure of D_{n+1} and D_n is a

measure of nD_{n-1}, and also that every measure of D_{n+1} and n is a measure of D_n. The inference from these two facts is not, as stated, that the G. C. M. of D_{n+1} and D_n must be that of D_n and D_{n-1}, but that the former G. C. M. can only differ from the latter by being multiplied by factors of the latter. Hence, instead of 2 being the G. C. M., we can only infer that the G. C. M. is some power of 2. It is easily seen, by calculating the first seven values of D, that 2 is not the G. C. M. To complete the investigation, assume that 2^m is the G. C. M. of D_{4m-2} and D_{4m-1}; and let

$$D_{4m-2} = A2^m, \quad D_{4m-1} = B2^m,$$

where one of the two quantities A, B is odd; then, writing M for *some multiple of*, we get

$$D_{4m} = (8m-1)\,D_{4m-1} - (4m-1)\,D_{4m-2} = M2^{m+2} + (A-B)\,2^m,$$
$$D_{4m+1} = (8m+1)\,D_{4m} - 4m\,D_{4m-1} = M2^{m+2} + (A-B)\,2^m,$$
$$D_{4m+2} = (8m+3)\,D_{4m+1} - (4m+1)\,D_{4m} = M2^{m+2} + 2\,(A-B)\,2^m$$
$$= \{M2 + (A-B)\}\,2^{m+1};$$
$$D_{4m+3} = (8m+5)\,D_{4m+2} - (4m+2)\,D_{4m+1} = M2^{m+2} + 8\,(A-B)\,2^m = M2^{m+2}.$$

Thus, if D_{4m-2}, D_{4m-1} are one an odd and the other an even multiple of 2^m, D_{4m}, D_{4m+1} are both odd multiples of 2^m; D_{4m+2} is an odd multiple of 2^{m+1}, and D_{4m+3} is a multiple of 2^{m+2}; and, for one value of m, D_{4m-2} is an odd multiple of 2^m, and D_{4m-1} is an even multiple of 2^m, namely, when $m = 1$; therefore D_{4m}, D_{4m+1}, D_{4m+2}, D_{4m+3} always have values of the form found above. Therefore the G. C. M. of D_{4m} and D_{4m+1} is 2^m, of D_{4m+1} and D_{4m+2} is 2^m, of D_{4m+2} and D_{4m+3} is 2^{m+1}, of D_{4m+3} and D_{4m+4} is 2^{m+1}; or, generally, of D_n and D_{n+1}, the G. C. M. is 2^k, where k is the greatest integer in $\frac{1}{4}(n+2)$.

6770. (By Professor WOLSTENHOLME, M.A.)—A parallelogram of minimum perimeter is inscribed in a given ellipse; prove that a conic can be found which cuts the ellipse at right angles at each of the angular points of the parallelogram, and that this conic passes through four fixed points (two real and two impossible).

I. *Solution by* G. F. WALKER, M.A.; F. BUDD, M.A.; *and others.*

The sides of the parallelogram must touch a confocal whose equation is found to be

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2+b^2},$$

and the polar reciprocal of this with respect to the ellipse is $x^2+y^2=a^2+b^2$ (a result which is evident geometrically); and the tangents to the ellipse at two adjacent points of the parallelogram are therefore at right angles.

Let the points on the ellipse be θ_1, θ_2; then we have

$$\frac{\cos\theta_1\cos\theta_2}{a^2} = -\frac{\sin\theta_1\sin\theta_2}{b^2} = \kappa \text{ suppose.}$$

The equation to any conic through the four points is of the form

$$\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+2\left(\frac{x}{a}\sin\theta_1-\frac{y}{b}\cos\theta_1\right)\left(\frac{x}{a}\sin\theta_2-\frac{y}{b}\cos\theta_2\right)=0.$$

If this cut the ellipse at right angles at the point θ_1, we have

$$\frac{\cos\theta_1}{a}\left[\lambda\frac{\cos\theta_1}{a}+\frac{\sin\theta_1}{a}\sin(\theta_2-\theta_1)\right]+\frac{\sin\theta_1}{b}\left[\lambda\frac{\sin\theta_1}{b}-\frac{\cos\theta_1}{b}\sin(\theta_2-\theta_1)\right]=0$$

$$\therefore\qquad \lambda\left(\frac{\sin^2\theta_1}{b^2}+\frac{\cos^2\theta_1}{a^2}\right)=\sin\theta_1\cos\theta_1\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\sin(\theta_2-\theta_1),$$

$$\lambda\sin\theta_2\cos\theta_2\left(\frac{\sin^2\theta_1}{b^2}+\frac{\cos^2\theta_1}{a^2}\right)$$

$$=\sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\sin(\theta_2-\theta_1),\quad\therefore\ \lambda\kappa\sin(\theta_2-\theta_1),$$

$$=-\kappa^2(a^2-b^2)\sin(\theta_2-\theta_1),\quad\therefore\ \lambda=-\kappa(a^2-b^2).$$

This is symmetrical with respect to θ_1 and θ_2, and therefore the conic cuts the given conic at right angles at θ_2 also.

The equation may be written

$$-\kappa(a^2-b^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+\left(\frac{a^2}{b^2}y^2-\frac{b^2}{a^2}x^2\right)\kappa-\frac{xy}{ab}\sin(\theta_1+\theta_2)=0,$$

which passes through the foci of the given conic.

II. *Solution by the* Proposer.

The sides of a parallelogram of minimum perimeter inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ must touch a confocal ellipse $\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}=1$; when λ will be determined by the equation

$$\frac{a^2-\lambda}{a^2}+\frac{b^2-\lambda}{b^2}=1,\quad\text{or } \lambda=\frac{a^2b^2}{a^2+b^2};$$

and the sides will touch the ellipse

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}.$$

The equation of two parallel sides may then be taken to be

$$\left(\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta\right)^2=\frac{a^2\cos^2\theta+b^2\sin\theta}{a^2+b^2};$$

and the equation of a conic through the corners of the parallelogram is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+k\left\{\left(\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta\right)^2-\frac{a^2\cos^2\theta+b^2\sin^2\theta}{a^2+b^2}\right\}=0.$$

One such conic is then $(k=-2)$

$$-\frac{x^2}{a^2}\cos2\theta+\frac{y^2}{b^2}\cos2\theta-\frac{2xy}{ab}\sin2\theta=1-2\,\frac{a^2\cos^2\theta+b^2\sin^2\theta}{a^2+b^2},$$

or

$$\frac{x^2}{a^2}+\frac{2xy}{ab}\tan2\theta-\frac{y^2}{b^2}=\frac{a^2-b^2}{a^2+b^2},$$

which cuts at right angles the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at all four common points. For at their common points

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{2xy}{ab}\tan 2\theta=\frac{a^2-b^2}{a^2+b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right);$$

and this will be found to coincide with the equation

$$\frac{x}{a^2}\left(\frac{x}{a^2}+\frac{y}{ab}\tan 2\theta\right)+\frac{y}{b^2}\left(\frac{x}{ab}\tan 2\theta-\frac{y}{b^2}\right)=0,$$

which is the condition that the tangents at a common point (x, y) may be at right angles.

This conic passes through the four fixed points

$$y=0,\ x^2=\frac{a^2(a^2-b^2)}{a^2+b^2};\quad x=0,\ y^2=\frac{b^2(b^2+a^2)}{b^2-a^2}.$$

6114. (By E. W. Symons, B.A.)—Prove that, in a spheric triangle,

$$\text{(i.) } \sec^2 A+\sec^2 B+\sec^2 C>12,\quad \text{(ii.) } \sec A\sec B\sec C>8;$$

and in a plane triangle, the triangles not being obtuse,

$$\text{(i.) } \tan A+\tan B+\tan C=\tan A\tan B\tan C>3\sqrt{3},$$

$$\text{(ii.) } \cot A+\cot B+\cot C>\sqrt{3},$$

$$\text{(iii.) } \tan B\tan C+\tan C\tan A+\tan A\tan B>9.$$

Solution by the Rev. T. R. Terry, M.A.; *the* Proposer; *and others.*

In a plane acute-angled triangle,

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C,$$

and $\quad \tan A+\tan B+\tan C>3(\tan A\tan B\tan C)^{\frac{1}{3}},$

therefore $\quad \tan A+\tan B+\tan C=\tan A\tan B\tan C>3\sqrt{3}$ .... (1),

Again, since $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$,

therefore $(\cot A+\cot B+\cot C)^2=3+\cot^2 A+\cot^2 B+\cot^2 C$
$\qquad -\cot A\cot B-\cot B\cot C-\cot C\cot A,$

therefore $\quad \cot A+\cot B+\cot C>\sqrt{3}$ .... (2).

From (1) and (2), $\tan A\tan B+\tan B\tan C+\tan C\tan A>9$,

also $\quad \sec^2 A+\sec^2 B+\sec^2 C>3+3(\tan A\tan B\tan C)^{\frac{2}{3}}$, *i.e.* >12,

and $\quad \sec^2 A\sec^2 B\sec^2 C=1+(\tan^2 A+\ldots)+(\tan^2 A\tan^2 B+\ldots)$
$\qquad +\tan^2 A\tan^2 B\tan^2 C$, *i.e.* >64,

therefore $\quad \sec A\sec B\sec C>8.$

Now, since these last two expressions increase with A, B, C, and as a plane triangle can always be drawn whose angles have the same ratio to one another as the angles of a spheric triangle, we see that, if the last two inequalities are true for the plane triangle, they are *a fortiori* true for the spherical one.

6475. (See Vol. XXXV., p. 21). II. *Solution by* W. J. C. Sharp, M.A.

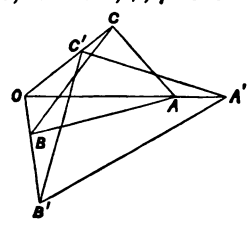
If $dy \div \{\phi(y, 1)\}^{\frac{1}{2}}$ be transformed by the proposed substitution, it becomes $$(U'V - UV')\, dx \div \{\phi(U, V)\}^{\frac{1}{2}},$$
and $\phi(U, V)$ may be determined so as to have four square factors
$$(x-\alpha)^2,\ (x-\beta)^2,\ (x-\gamma)^2, \text{ and } (x-\delta)^2.$$
The conditions for this will determine the ratios of the coefficients of $\phi(y, 1)$; and the product of $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ will divide $U'V - UV'$, which is itself of the fourth degree in x (see Cayley's *Elliptic Functions*, p. 163); therefore
$$\frac{(U'V-UV')\,dx}{\{\phi(U, V)\}^{\frac{1}{2}}} = \frac{dx}{M\{f(x)\}^{\frac{1}{2}}},$$
where $f(x)$ is the quartic function of x, which is obtained by dividing $\phi(U, V)$ by $(x-\alpha)^2(x-\beta)^2(x-\gamma)^2(x-\delta)^2$.

[This theorem shows that an elliptic integral can always be found which will be transformable by any given transformation of the third order. It appears from the above, however, that it is not possible to find such an integral for a given transformation of a higher order.]

5925. (By E. W. Symons, M.A.)—If A′, B′, C′ be any three points on the edges OA, OB, OC of a tetrahedron; prove that the cosine of the angle between the planes ABC, A′B′C′ is
$$\{\Delta_1\Delta_1' + \Delta_2\Delta_2' + \Delta_3\Delta_3' - (\Delta_2\Delta_3' + \Delta_2'\Delta_3)\cos A' - \dots\} \div \Delta\Delta',$$
where Δ is the area of the triangle ABC, Δ_1 of OBC, ... &c., and A′ the angle between the planes OBC, OCA, &c.

Solution by the Proposer; A. L. Selby, M.A.; *and others.*

Let $OA = a$, $OB = b$, $OC = c$, $OA' = a'$, ...; and let α, β, γ be unit vectors along the axes; then
$$(AB) = b\beta - a\alpha,\quad (AC) = c\gamma - a\alpha,$$
and $\dfrac{V.[(AB).(AC)]}{2\Delta}$ is unit vector perpendicular to plane ABC; so $\dfrac{V[(A'B').(A'C')]}{2\Delta'}$ is unit vector perpendicular to plane A′B′C′, and
$$\pm 4\Delta\Delta'.\cos\theta$$
$$= S.[V.(AB)(AC)][V.(A'B').(A'C')],$$



$\equiv$ [by well known formulæ, see KELLAND's *Quaternions*, Art. 70, (16)]

$$S.[(AB).(A'C')].S[(AC).(A'B')]-S[(AB).(A'B')].S[(AC).(A'C')]$$

$$= S(b\beta-a\alpha)(c'\gamma-a'\alpha).S(c\gamma-a\alpha)(b'\beta-a'\alpha)$$

$$-S(b\beta-a\alpha)(b'\beta-a'\alpha).S(c\gamma-a\alpha)(c'\gamma-a'\alpha)$$

$$\equiv (bc'\cos\lambda-c'a\cos\mu-a'b\cos\nu+aa')(b'c\cos\lambda-ca'\cos\mu-ab'\cos\nu+aa')$$

$$-[aa'+bb'-(ab'+a'b).\cos\nu][aa'+cc'-(ac'+a'c).\cos\mu]$$

$$\equiv bc.b'c'.\sin^2\lambda+\ldots+\ldots+aa'(bc'+b'c)(\cos\mu\cos\nu-\cos\lambda)+\ldots$$

$$\equiv 4.\Delta_1.\Delta_1'+\ldots-4(\Delta_2\Delta_3'+\Delta_2'\Delta_3)\cos A;$$ therefore, &c.

6352. (By W. H. WALENN, Mem. Phys. Soc.)—Check the calculation

$$V' = \frac{1}{3}.\frac{600\times20-384\times16}{20-16}\times36 = 17568$$

by casting out the elevens, the form being preserved, and no quotients of the divisor 11 being known or used in the process.

Solution by the PROPOSER.

The principles that apply to the solution of this question are treated of in the following papers upon unitation, published in the *Philosophical Magazine*:—II. July, 1873; III. May, 1875; VII. Nov. 1877, (par. 27); "Checking calculations," Jan. 1880; and IX. Feb. 1880, (par. 39).

The subtraction of the sum of the digits in odd places, from the sum of the digits in even places, applied to each value or datum, in the same equational form gives $U_{11}V' = U_{11}\left\{\frac{1}{3}.\frac{6\times9-10\times5}{9-5}\times3\right\}$, writing $U_{11}V'$ for "unitate of V′ to the base 11," or "the remainder to the division of V′ by 11," regarded as a separate function that expresses the number of units that a given number (V′) is in excess of being exactly divisible by 11.

All values of $U_{11}x^{-1}$, excepting that of $U_{11}11^{-1}$ and (in general) $U_{11}(11m)^{-1}$, are identical with the series $U_{11}x^9$, as the series $U_{11}x^n$ repeats after every 10 terms. Therefore

$$U_{11}V' = U_{11}\{4.3(10-6)\times3\} = U_{11}\{3(10-6)\} = U_{11}\{3.4\} = 1,$$

and $$U_{11}17568 = 1.$$

The correctness of $V' = 17568$ is thus ascertained within a high degree of probability. The value of V′ is taken from J. PRYDE's *Practical Mathematics* (CHAMBERS's *Educational Course*), p. 222.

6424. (By J. W. RUSSELL, M.A.)—In any triangle, prove that

$$\tfrac{27}{4} > (\sin A+\sin B+\sin C)^2 > 6\sqrt{3}\sin A\sin B\sin C.$$

Solution by H. L. ORCHARD, M.A.; D. EDWARDES; *and others.*

Let r_1, r_2, r_3 be the escribed radii; then we have

$$(r_1-r_2)^2+(r_2-r_3)^2+(r_3-r_1)^2 = 2\left\{(r+4\text{R})^2-3s^2\right\},$$

therefore $\qquad r+4\text{R}>s\sqrt{3}$ and $r<\frac{1}{2}\text{R}$,

therefore, *à fortiori*, $\qquad \frac{9}{2}\text{R}>s\sqrt{3}$, or $27\text{R}^2>4s$;

that is, $\qquad \frac{27}{4}>(\sin\text{A}+\sin\text{B}+\sin\text{C})^2$.

Again, $\qquad (a-b)^2+(b-c^2)+(c-a)^2 = 2(s^2-3r^2-12\text{R}r)$,

therefore $\qquad s^2>3r(r+4\text{R})$ and $(r<\frac{1}{2}\text{R})$,

therefore $\qquad s^2>27r^2$, or $s^2>3\sqrt{3}\Delta$;

that is, $4s^2>3\sqrt{3}\dfrac{abc}{\text{R}}$, or $(\sin\text{A}+\sin\text{B}+\sin\text{C})^2>6\sqrt{3}\sin\text{A}\sin\text{B}\sin\text{C}$.

[This is an immediate consequence of the well known theorem, that of all triangles that can be inscribed in a given circle, the equilateral has the greatest perimeter; and of all the circumscribed, the least perimeter,—a theorem whereof the above solution furnishes a trigonometrical proof.]

6242 & 6274. (By Prof. ROSANES.)—(6242) Prove that (1) in the plane of three given conics A, B, C there are three sets of points a, b, c, such that A_b, B_c, C_a are respectively identical with B_a, C_b, A_c; where, generally, P_q denotes the polar of a point q with respect to a conic P; (2) of the three triangles formed respectively by the three points a, the three points b, and the three points c, the first is in perspective with the second, the second with the third, and the third with the first; (3) nine points having the last-named property being given, three conics, related to them in the manner above described, are uniquely determinable.

(6274). The coordinates g_1, g_2, g_3, g_4, g_5, g_6 of a line G being given, those of its conjugate G′, relative to a surface F of the second order, are determined by six equations of the form

$$\rho g_i' = \sum_{s=0}^{s=6} \text{C}_{i,s}\cdot g_s \quad (i=1,2,\ldots 6),$$

where the $\text{C}_{i,s}$ denote minors of the 2nd degree, of the determinant of F, and ρ a constant factor. According to this, the determination of a *self*-conjugate line leads to a sextic equation in ρ. How is this to be reconciled with the fact that every generator of F is self-conjugate?

Solution by W. J. C. SHARP, M.A.

(6242). If the self-conjugate triangle of B and C be taken as triangle of reference, so that $\text{A}\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$,

$$\text{B}\equiv a'x^2+b'y^2+c'z^2=0,\ \text{and}\ \text{C}\equiv a''x^2+b''y^2+c''z^2=0,$$

and if a, b, c be (α,β,γ), (x,y,z), (ξ,η,ζ) respectively;

$$\frac{ax+hy+gz}{a'\alpha}=\frac{hx+by+fz}{b'\beta}=\frac{gx+fy+cz}{c'\gamma},\quad \frac{a'\xi}{a''x}=\frac{b'\eta}{b''y}=\frac{c'\zeta}{c''z},$$

$$\frac{a\xi+h\eta+g\zeta}{a''\alpha}=\frac{h\xi+b\eta+f\zeta}{b''\beta}=\frac{g\xi+f\eta+c\zeta}{c''\gamma};$$

therefore $\dfrac{a'\left(\dfrac{aa''}{a'}x+\dfrac{hb''}{b'}y+\dfrac{gc''}{c'}z\right)}{a''(ax+hy+gz)} = \dfrac{b'\left(\dfrac{ha''}{a'}x+\dfrac{bb''}{b'}y+\dfrac{fc''}{c'}z\right)}{b''(hx+by+fz)}$

$$= \frac{c'\left(\dfrac{ga''}{a'}x+\dfrac{fb''}{b'}y+\dfrac{cc''}{c'}z\right)}{c''(gx+fy+cz)} \quad \text{.........(1).}$$

Equations which determine three points b, similarly three points c, and three points a, are determined. If $\lambda x+\mu y+\nu z=0$ be the equation to the line joining corresponding b and c points (x_1, y_1, z_1) and (ξ_1, η_1, ζ_1),

$$\lambda : \mu : \nu = \frac{1}{x_1}\left(\frac{b''}{b'}-\frac{c''}{c'}\right) : \frac{1}{y_1}\left(\frac{c''}{c'}-\frac{a''}{a'}\right) : \frac{1}{z_1}\left(\frac{a''}{a'}-\frac{b''}{b'}\right),$$

where (x_1, y_1, z_1) satisfies (1). Hence it appears that $\dfrac{g\mu-h\nu}{g-h}=\dfrac{h\nu-f\lambda}{h-f}$. Consequently the lines joining corresponding b and c points meet in a point, and the triangles formed by the b and c points are in perspective; and, by symmetry, so are those by the c and a points, and by the a and b points. Again, if the full forms be used for $B=0$ and $C=0$, the equations expressing the identities $A_{b_1}=B_{a_1}$, &c., for the nine-points a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3, will determine ratios of the coefficients of each equation uniquely.

(6274). If (x', y', z'), (x'', y'', z'') be two points on the line whose metrical coordinates are $g_1, g_2, g_3, g_4, g_5, g_6$; these are proportional to

$$x'-x'' : y'-y'' : z'-z'' : y'z''-y''z' : z'x''-z''x' : x'y''-x''y'.$$

And the equations to the conjugate with respect to

$$F \equiv ax^2+by^2+cz^2+2lyz+2mzx+2nxy+2px+2qy+2rz+d=0$$

are $\quad F'_1x+F'_2y+F'_3z+F'_4=0$, and $F''_1x+F''_2y+F''_3z+F''_4=0$,

where F_1, F_2, &c. are used to denote $\dfrac{dF}{dx}$, $\dfrac{dF}{dy}$, &c.; therefore

$$g'_1 : g'_2 : g'_3 : g'_4 : g'_5 : g'_6 = F'_2F''_3-F''_2F'_3 : F'_3F''_1-F''_3F'_1 : F'_1F''_2-F''_1F'_2$$
$$: F''_4F'_1-F'_4F''_1 : F''_4F'_2-F'_4F''_2 : F''_4F'_3-F'_4F''_3,$$

and $\quad F'_2F''_3-F''_2F'_3=(nr-qm)\,g_1+(br-ql)\,g_2+(rl-qc)\,g_3$
$$+(bc-l^2)\,g_4+(lm-nc)\,g_5+(nl-bm)\,g_6, \text{ \&c. \&c.,}$$

which proves the equation given.

The reason why there is no limitation to the number of self-conjugate lines is, that these equations are not independent. In fact, ρ is a constant, as appears from the form taken by these equations if the centre and axes be taken as origin and axes of coordinates; then

$$\rho g_1=bcg_4,\ \rho g_2=acg_5,\ \rho g_3=abg_6,\ \rho g_4=adg_1,\ \rho g_5=bdg_2,\ \rho g_6=cdg_3,$$

and $\rho^2=abcd=\Delta$. Also, from the general forms, it appears that

$$F'_1g'+F'_2g'_2+F'_3g'_3=0 \text{ and } F'_1g'_1+F''_2g'_2+F''_3g'_3=0,$$

and for a self-conjugate line $F'_1g_1+F'_2g_2+F'_3g_3=0$; and if one point is on the surface, all points are so, and the line is a generator.

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